PARTITIONING AN INTEGER

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Abstract. We present a generating function for the number of ways of partitioning a positive integer n into distinct positive integers $\lambda_1 > \ldots > \lambda_r > 0$ such that $\lambda_1 \leq k$, for fixed k. A way of rapidly tabulating the number of such partitions for fixed m is then given. Finally, we write down a generating function for the number of partitions of n for fixed k and fixed r.

Definition 1. A *partition* of a positive integer n is a finite nonincreasing sequence of positive integers $\langle \lambda_1, \lambda_2, \ldots, \lambda_r \rangle$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are called the *parts* of the partition.

Definition 2. Let $\mathcal{P}$ denote the set of all partitions. Let $\mathcal{D} \subset \mathcal{P}$ be the set of all partitions with distinct parts. Let $\mathcal{D}_k \subset \mathcal{D}$ be the set of all partitions with distinct parts such that $\lambda_1 \leq k$.

Definition 3. Let $p(S, n)$ denote the number of partitions of n that belong to a subset S of the set $\mathcal{P}$ of all partitions.

Some immediate observations are that $p(\mathcal{D}_n, n) = p(\mathcal{D}_{n-1}, n) + 1$ and that

$$p(\mathcal{D}_m, n) = \begin{cases} 0 & \text{if } m(m+1) < 2n \\ p(\mathcal{D}, n) & \text{if } m \geq n \end{cases}$$

Example 1. The distinct partitions of $n = 10$ are $\langle 4,3,2,1 \rangle$, $\langle 5,3,2 \rangle$, $\langle 5,4,1 \rangle$, $\langle 6,3,1 \rangle$, $\langle 7,2,1 \rangle$, $\langle 6,4 \rangle$, $\langle 7,3 \rangle$, $\langle 8,2 \rangle$, $\langle 9,1 \rangle$ and $\langle 10 \rangle$. Hence $p(\mathcal{D}, 10) = 10$ and the $p(\mathcal{D}_i, 10)$ are given by

i	≤ 3	4	5	6	7	8	9	≥ 10
$p(\mathcal{D}_i, 10)$	0	1	3	5	7	8	9	10

The following interesting results concerning $p(\mathcal{D}, n)$ are well known (see (Hardy & Wright 1979) ppg.276-7).

Theorem 1. *The generating function of* $p(\mathcal{D}, n)$ *is given by*

$$\prod_{i=1}^{\infty} (1 + x^i)$$

Theorem 2. *Let* $\mathcal{O}$ *be the set of all partitions whose parts are odd integers. Then* $p(\mathcal{D}, n) = p(\mathcal{O}, n)$.

Let $\mathcal{O}_k \subset \mathcal{O}$ be the set of all partitions with odd parts such that $\lambda_1 \leq k$. Then $\mathcal{O}_{2k} = \mathcal{O}_{2k+1}$ and hence $p(\mathcal{O}_{2k}, n) = p(\mathcal{O}_{2k+1}, n)$. Thus the obvious analogue of Theorem 2 is false.

We shall now prove a result, similar to Theorem 1, giving the generating function of $p(\mathcal{D}_k, n)$.

Theorem 3. *Given some fixed* $k \in \mathbb{N}$ *the generating function of* $p(\mathcal{D}_k, n)$ *is given by*

$$\prod_{i=1}^{k} (1 + x^i)$$

Proof. This is essentially immediate since the product (and hence all the sums involved) is finite. We simply observe the following:

$$\begin{aligned}\prod_{i=1}^{k}(1+x^i) &= \sum_{a_1\in\{0,1\}}\cdots\sum_{a_k\in\{0,1\}} x^{\sum_{i=1}^{k} a_i i}\\ &= \sum_{j=1}^{k} p(\mathcal{D}_k,j)x^j\end{aligned}$$

□

A consequence of Theorem 3 is that it allows us to tabulate the values of $p(\mathcal{D}_k,n)$ by using induction on column vectors.
Let us fix k and let $\triangle_k = \frac{k(k+1)}{2}+1$.
Let $A_0 = \begin{pmatrix}1\\1\\0\\\vdots\\0\end{pmatrix} \in M_{\triangle_k\times 1}$ and let $S = \begin{pmatrix}0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0\end{pmatrix} \in M_{\triangle_k}$.
Make the recursive definition

$$A_{i+1} = A_i + S^{i+1}A_i$$

Then the value of $p(\mathcal{D}_k,n)$ is given by the $(n+1)^{th}$-entry in the column vector A_k (or is 0 if $n \geq \triangle_k$).

Theorem 4. *Using notation as above,* $p(\mathcal{D}_k,n) = p(\mathcal{D}_k,\triangle_k - n - 1)$ *for all* n *and* $k \geq 1$.

Proof. It is sufficient to show that A_k is symmetrical about the $\left\lfloor\frac{\triangle_k}{2}\right\rfloor^{th}$-entry. We proceed by induction on k. The result is true for $k=1$. Assume it is true for $k=m$. Then A_{m+1} is constructed by adding A_m to a copy of A_m whose entries have been shifted down $m+1$ places. Hence the resulting vector will be symmetrical about

$$\begin{aligned}\left\lfloor\frac{\triangle_m}{2}\right\rfloor + \left\lfloor\frac{m+1}{2}\right\rfloor &= \left\lfloor\frac{m(m+1)+2}{4}\right\rfloor + \left\lfloor\frac{m+1}{2}\right\rfloor\\ &= \left\lfloor\frac{(m+1)(m+2)+2}{4}\right\rfloor\\ &= \left\lfloor\frac{\triangle_{m+1}}{2}\right\rfloor\end{aligned}$$

□

Corollary 1. $p(\mathcal{D}_k,n) = p(\mathcal{D}_{k-1},n) + p(\mathcal{D}_{k-1},n-k)$.

Proof. The result follows immediately from the observation that

$$\prod_{i=1}^{k}(1+x^i) - \prod_{i=1}^{k-1}(1+x^i) = x^k\prod_{i=1}^{k-1}(1+x^i).$$

□

Example 2. Using these results it is possible to swiftly calculate (by hand) the values of $p(\mathcal{D}_k,n)$ for $k \leq 10$ and $n \leq \left\lfloor\frac{\triangle_k}{2}\right\rfloor$

n \ k	1	2	3	4	5	6	7	8	9	10
1	1	...								
2		1	...							
3		1	2	...						
4			1	2	...					
5			1	2	3	...				
6			1	2	3	4	...			
7				2	3	4	5	...		
8				1	3	4	5	6	...	
9				1	3	5	6	7	8	...
10				1	3	5	7	8	9	10
11					2	5	7	9	10	11
12					2	5	8	10	12	13
13					1	4	8	11	13	15
14					1	4	8	12	15	17
15					1	4	8	13	17	20
16						3	8	13	18	22
17						2	7	13	19	24
18						2	7	14	21	27
19						1	6	13	21	29
20						1	5	13	22	31
21						1	5	13	23	33
22							4	12	23	35
23							3	11	23	36
24							2	10	23	38
25							2	9	22	39
26							1	8	22	39
27							1	7	21	40

The C source code for a computer program which uses the above technique to calculate the values of $p(\mathcal{D}_k, n)$ for all n and arbritrary k is available from

`http://www.maths.bath.ac.uk/~mapamk/code/PartGenFun.c`

Definition 4. Let $\mathcal{D}_{k,m} \subset \mathcal{D}_k$ be the set of all partitions with distinct parts such that $\lambda_1 \leq k$ and $r = m$ (with notation as in Definition 1).

Definition 5. Let $S_{k,m}$ be the set

$$S_{k,m} := \left\{(a_1, \ldots, a_m) \in \{1, \ldots, k\}^m \mid a_1 < \ldots < a_m\right\}.$$

For $\sigma = (a_1, \ldots, a_m) \in S_{k,m}$ we regard it as a map $\sigma : \mathbb{N} \to \mathbb{N}$ via

$$\sigma i = \begin{cases} a_i & \text{if } 1 \leq i \leq m \\ i & \text{otherwise} \end{cases}$$

Clearly $|S_{k,m}| = \binom{k}{m}$.

Theorem 5. *Given some fixed $k, m \in \mathbb{N}$ with $m \leq k$, the generating function of*

$$p(\mathcal{D}_{k,m}, n) - p(\mathcal{D}_{k-1,m}, n)$$

is given by

$$\sum_{\sigma \in S_{k-1,m-1}} \prod_{i=1}^{m-1} (x^k + x^{\sigma i}).$$

It is an immediate consequence that that generating function for $p(\mathcal{D}_{k,m}, n)$ is given by

$$\sum_{j=m}^{k} \sum_{\sigma \in S_{j-1,m-1}} \prod_{i=1}^{m} (x^j + x^{\sigma i}).$$

References

Hardy, G. H. & Wright, E. M. (1979), *An Introduction to the Theory of Numbers*, Oxford Science Publications, fifth edn, Oxford University Press.