COMPUTATIONAL COMMUTATIVE ALGEBRA MIDTERM

Choose one of the following topics. Write a report (it need not be typed) in which you explore the topic, and present a 20 minute presentation to the class. The report and presentation will be submitted during the first week after the March break. Please don't hesitate to see me if you have any questions, or need help.

(1) What have Göbner bases got to do with robotics ([3, Chapter 6, §1–3])?

(2) It's fun to study curves in the plane (either in $\mathbb{R}^2$ or in $\mathbb{C}^2$). Given a polynomial $f \in k[x, y]$, the *singular locus* is defined by the vanishing of the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$. You can calculate a Gröbner basis in order to find the singular points of the variety $\mathbb{V}(f)$ defined by f.

Do this calculation for the lemniscate. Can you generalise the definition of the lemniscate in any way? Look for examples with singularities. Look at the singular locus for other interesting curves (for a few examples, consult [13]). What about the intersection of two surfaces in $\mathbb{R}^3$? Your report should contain plenty of examples and sketches. For details about singularities on curves, see [5].

(3) David Hilbert was perhaps one of the most influential mathematicians of the last century. In this course we have proved the Hilbert Basis Theorem, which is fundamental to our understanding of ideals in polynomial rings. Write a short essay on Hilbert's life and his many mathematical contributions. Try to include some other important results. Make sure that you explain what Hilbert's famous twenty-three problems are, and whether they have been solved.

An excellent place to start your research is [6]. Wikipedia also has a good article on Hilbert's problems ([14]) which you can use to find references to further material.

(4) Emmy Noether was an astonishing mathematician, overcoming the prejudices of the time to contribute a staggering quantity of research. Write a short essay on her life. You should include a brief explanation of the concepts of Noetherian rings, the ascending chain condition, and any other ideas from commutative algebra which you feel are relevant to this course.

The mathematics is covered in any introductory book on commutative algebra (for example, [2, 9, 11]). For a brief biography, consult [7].

http://erdos.math.unb.ca/∼kasprzyk/
kasprzyk@unb.ca.

(5) Implement the division algorithm in $k[x_1, \ldots, x_n]$ using Maple. Allow the user to specify which monomial order to use. This might be deceptively difficult, or it might turn out to be very easy. If it's too easy, consider implementing your own version of other algorithms we've seen in the course (such as finding a Gröbner basis, or taking a Gröbner basis and turning it into a reduced Gröbner basis).

(6) We know that for an arbitrary ideal $I \subset k[x_1, \ldots, x_n]$ we have the inclusion $I \subset \mathbb{I}(\mathbb{V}(I))$. Under what conditions is this an equality? How is the ideal $\mathbb{I}(\mathbb{V}(I))$ related to I?

You should explore a little more of affine algebraic geometry. What is a coordinate ring? What is the Nullstellensatz? How is the Zariski topology defined? Good sources of information are [3, 5, 8, 9, 12].

(7) Similar to (6): What is projective space $\mathbb{P}^n$? Describe a little of projective algebraic geometry. Why do people work in $\mathbb{P}^n$? You may wish to consult [5, 8, 9, 10, 12].

(8) A *graph* is a collection of vertices connected by edges (this use of the word "graph" it totally distinct from the more familiar meaning). Many problems can be expressed as a graph, where the problem reduces to assigning *colours* to each of the vertices such that no two vertices connected by an edge have the same colour. If the number of colours you can use is limited, is it possible to colour the vertices in such a fashion? Is there a systematic way of finding a solution? This is called the *graph colouring problem*.

Gröbner bases offer a way of solving the graph colouring problem. You should explain how this is done. Make sure that you explain the theory, and give plenty of examples. Details can be found in [1]. An entertaining (if somewhat impractical) application is in solving Sudoku puzzles ([4]).

(9) Given a system of polynomial equations over $\mathbb{C}$, a combination of Gröbner bases and techniques from numerical methods can be used to find all possible solutions. For example, [3, Theorem 6, pg. 234] can be used to determine whether there are finitely many solutions. Numerical methods can then be used to solve for one variable at time from a Gröbner basis.

You should write code in *Maple* (or some other language) to implement [3, Theorem 6, pg. 234] and, when the number of solutions are finite, calculate all possible answers. Make sure that you test your code on plenty of examples.

(10) Recall that we have refrained from giving a definition of what we mean by the *dimension* of an affine variety $\mathbb{V}(I)$. Our intuitive idea of dimension certainly seems to be correct most of the time, however we have seen examples where it fails. If I is a monomial ideal, however, developing a rigourous understanding of what we mean by dimension is not too difficult.

You should explain how we define $\dim \mathbb{V}(I)$ when I is a monomial ideal, and give examples to illustrate this definition. Consult [3, Chapter 9, §1–2]. If you have time, you may also wish to look at the *Hilbert function*.

References

1. William W. Adams and Philippe Loustaunau, *An introduction to Gröbner bases*, Graduate Studies in Mathematics, vol. 3, American Mathematical Society, Providence, RI, 1994.
2. M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.
3. David Cox, John Little, and Donal O'Shea, *Ideals, varieties, and algorithms*, third ed., Undergraduate Texts in Mathematics, Springer, New York, 2007, An introduction to computational algebraic geometry and commutative algebra.
4. Jesús Gago-Vargas, Isabel Hartillo-Hermoso, Jorge Martín-Morales, and José María Ucha-Enríquez, *Sudokus and Gröbner bases: not only a divertimento*, Computer algebra in scientific computing, Lecture Notes in Comput. Sci., vol. 4194, Springer, Berlin, 2006, available online at `http://www.ricam.oeaw.ac.at/Groebner-Bases-Bibliography/gbbib_files/publication_1180.pdf`, pp. 155–165.
5. Frances Kirwan, *Complex algebraic curves*, London Mathematical Society Student Texts, vol. 23, Cambridge University Press, Cambridge, 1992.
6. J. J. O'Connor and E. F. Robertson, *David Hilbert biography*, online, access via `http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Hilbert.html`.
7. ———, *Emmy Noether biography*, online, access via `http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Noether_Emmy.html`.
8. Miles Reid, *Undergraduate algebraic geometry*, London Mathematical Society Student Texts, vol. 12, Cambridge University Press, Cambridge, 1988.
9. ———, *Undergraduate commutative algebra*, London Mathematical Society Student Texts, vol. 29, Cambridge University Press, Cambridge, 1995.
10. Igor R. Shafarevich, *Basic algebraic geometry. 1*, second ed., Springer-Verlag, Berlin, 1994, Varieties in projective space, Translated from the 1988 Russian edition and with notes by Miles Reid.
11. R. Y. Sharp, *Steps in commutative algebra*, second ed., London Mathematical Society Student Texts, vol. 51, Cambridge University Press, Cambridge, 2000.
12. Karen E. Smith, Lauri Kahanpää, Pekka Kekäläinen, and William Traves, *An invitation to algebraic geometry*, Universitext, Springer-Verlag, New York, 2000.
13. Unknown, *Famous curves index*, online, access via `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Curves.html`.
14. Wikipedia, *Hilbert's problems*, online, access via `http://en.wikipedia.org/wiki/Hilbert's_problems`.