MATH3353

COMPUTATIONAL COMMUTATIVE ALGEBRA
APRIL 2008

This exam consists of five questions, each worth 20 marks. Please attempt all the questions. This is a take–home, open–book exam. Unless stated otherwise, you may use a computational algebra package; you should print out a copy of your calculations and submit them along with your solutions.

1. State clearly the definition of *Gröbner basis*, and the definition of *S-polynomial*. Explain how the S-polynomial is used when calculating a Gröbner basis.

Without the aid of a computer, find a Gröbner basis for the ideal:

$$(y^2 - x, z^4 - y^2).$$

Be sure that you indicate which monomial order you are using, and that you explain how you know that your answer is a Gröbner basis,

Is your Gröbner basis minimal? Is it reduced? If not, adjust it so that it is.

2. (i) Find all possible solutions to the system of equations:

$$\begin{cases} x^{10} - 22x^6 + 51x^4 - 48x^2 + 18 = -18y, \\ x^{10} - 22x^6 + 51x^4 - 30x^2 + 18 = 18z, \\ x^{12} - 9x^{10} + 32x^8 - 57x^6 + 51x^4 - 18x^2 = 0. \end{cases}$$

If appropriate, write your solutions in terms of ζ and ζ^2, where $1, \zeta$ and ζ^2 are the three cube roots of unity.

(ii) Find a parameterisation for the curve:

$$\mathbb{V}(x^2 z - y^2, yx - x + 1) \subset \mathbb{R}^3.$$

What is the image of its projection along the x-axis onto the (y, z)-plane? In the image of this projection, explain what's happening at the point $(1, 0)$.

3. State the definition of *monomial ideal*. Prove that if

$$I = (x^\alpha \mid \alpha \in A)$$

is a monomial ideal, then a monomial $x^\beta \in I$ if and only if x^β is divisible by x^α for some $\alpha \in A$.

Sketch a picture of the monomials contained in the ideal:

$$J = (x^6, x^5y, x^3y^2, x^2y^3, y^5) \subset k[x, y].$$

If we perform the division algorithm on some $f \in k[x, y]$ using the generators of J as divisors, write down explicitly which terms can appear in the remainder. Making sure that you state which monomial order you are using, calculate the remainder when $f = 2x^3 + x^7y^2 + 3x^2y + y^6$.

4. (i) Let $I = (f_1, \ldots, f_r)$ be an ideal in $\mathbb{C}[x_1, \ldots, x_n]$. Prove that $f \in \sqrt{I}$ if and only if $1 \in (f_1, \ldots, f_r, 1 - wf) \subset \mathbb{C}[x_1, \ldots, x_n, w]$.

(ii) If $I = (xy^2 - 2y^2, x^4 - 2x^2 + 1) \subset \mathbb{C}[x, y]$ and $f = y - x^2 + 1$, is $f \in \sqrt{I}$? If so, what is the smallest power $m > 0$ of f such that $f^m \in I$?

(iii) Let $I = ((x^7 - x^6 - 2x^5 + 2x^4 + x^3 - x^2)y^2 + (2x^5 - 4x^4 + 4x^2 - 2x)y^3 + (x^3 - 3x^2 + 3x - 1)y^4)$ be a principal ideal in $\mathbb{C}[x, y]$. Find $\sqrt{I}$.

5. Let $f \in k[x_1, \ldots, x_n]$ be a polynomial. We say that f has a *singularity* at the point $(a_1, \ldots, a_n) \in k^n$ if its partial derivatives all vanish at that point; i.e. if:

$$\frac{\partial f}{\partial x_1}(a_1, \ldots, a_n) = \ldots = \frac{\partial f}{\partial x_n}(a_1, \ldots, a_n) = 0.$$

(i) Let $f = (x^4 + y^4)^2 - x^2y^2 \in \mathbb{R}[x, y]$. By performing a suitable Gröbner basis calculation, show that $\mathbb{V}(f, \partial f/\partial x, \partial f/\partial y) = \{(0, 0)\}$. Hence or otherwise conclude that $f = 0$ has only one singular point, at the origin.

(ii) Show that $g = 2xy^5 + 5y^2z^4 - 3x^2z^4$ has two lines of singularities. What are they?