1. Let $f : [0, 2] \to \mathbb{R}$ be given by $f(x) = x^3 + x - 1$. This is continuous on the given interval and, since $f$ is a polynomial, differentiable on $(0, 2)$. Hence it satisfies the conditions of the Mean Value Theorem.

The Mean Value Theorem tells us that there exists some $c \in (0, 2)$ such that:

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{9 - (-1)}{2} = 5.$$ 

Since $f'(x) = 3x^2 + 1$, we see that $3c^2 + 1 = 5$ and so $c = 2/\sqrt{3}$.

2. Proposition. Let $f$ and $g$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Then $f(b) < g(b)$.

Proof. Let $h = f - g$. Then, since both $f$ and $g$ are continuous on $[a, b]$, $h$ is continuous on $[a, b]$. Because $f$ and $g$ are differentiable on $(a, b)$, so $h$ is differentiable on $(a, b)$. Hence $h$ satisfied the conditions on the Mean Value Theorem, and there exists some $c \in (a, b)$ such that:

$$h'(c) = \frac{h(b) - h(a)}{b - a} \Rightarrow f'(c) - g'(c) = \frac{f(b) - g(b) - f(a) + g(a)}{b - a}.$$ 

By hypothesis $f'(c) < g'(c)$, and so $f'(c) - g'(c) < 0$. Since $f(a) = g(a)$ we obtain that:

$$\frac{f(b) - g(b)}{b - a} < 0 \Rightarrow f(b) - g(b) < 0 \Rightarrow f(b) < g(b),$$

as required. $\square$
3. (i) $f$ is increasing when the derivative $f'$ is positive. This occurs in the intervals $(2, 4)$ and $(6, 9)$.

(ii) $f$ has a local minimum when $f'$ changes from negative sign to positive sign. This occurs at $x = 2$ and $x = 6$. There is a local maximum when $f'$ changes from positive sign to negative sign. This occurs at $x = 4$. We must also consider the endpoints $x = 0$ and $x = 9$. Close to $x = 0$ the gradient is negative, hence we have a local maximum there. Close to $x = 9$ the gradient is positive, and we see that this is also a local maximum.

(iii) By the Concavity Test, $f$ is concave upwards in the regions where $f''$ is positive. Thus we require $f'$ to have a positive gradient. This occurs in the intervals $(1, 3), (5, 7)$ and $(8, 9)$. $f$ is concave downwards when $f''$ is negative. We thus require $f'$ to have negative gradient. This occurs in the intervals $[0, 1), (3, 5)$ and $(7, 8)$.

(iv) By definition a point of inflection is when $f$ swaps from being concave upwards to concave downwards, or vice versa. This occurs when $x = 1, x = 3, x = 5, x = 7, and x = 8$.

4. Let $B(x) = 3x^{2/3} - x$. Then $B'(x) = 2x^{-1/3} - 1$ and $B''(x) = -(2/3)x^{-4/3}$.

(i) $B$ is increasing when $B'$ is positive. This occurs when $2x^{-1/3} - 1 > 0$; i.e. when $2 > \sqrt[3]{x}$. Hence when $x < 8$. $B$ is decreasing when $B'$ is negative. We see that this occurs when $x > 8$.

(ii) Local maxima or minima occur when $B'$ changes sign, or at the boundary of the domain on which the function is defined. $B'$ changes sign once, when $x = 8$, from positive to negative. This implies that $B$ has a local maximum at the point $(8, 4)$. Since $B$ is defined only on $[0, \infty)$, we need to consider the point $x = 0$. Close to $x = 0 B$ has positive gradient. Hence there is a local minimum at the origin $(0, 0)$.

(iii) By the Concavity Test, $B$ is concave upwards in the regions where $B''$ is positive, and is concave downwards when $B''$ is negative. Hence $B$ is concave upwards when $-(2/3)x^{-4/3} > 0$, but since $x \geq 0$ this never occurs. On the other hand, $-(2/3)x^{-4/3} < 0$ for all $x \geq 0$, and so $B$ is concave downwards for all points in its domain.

By definition a point of inflection is when $B$ swaps from being concave upwards to concave downwards, or vice versa. Since $B$ is never concave upwards, there can be no points of inflection.

(iv) Sketching the graph is easy.
5. Consider \( y = \frac{x^2 - 2}{x^4} \). This has domain \( \mathbb{R} \setminus \{0\} \). Since:

\[
\frac{(-x)^2 - 2}{(-x)^4} = \frac{x^2 - 2}{x^4},
\]

the function is even. Hence the graph is symmetric about the \( y \)-axis.

(i) The roots \( y = 0 \) occur when \( x^2 - 2 = 0 \); i.e. when \( x = \pm\sqrt{2} \).

(ii) We consider the behaviour of the graph close to \( x = 0 \):

\[
\lim_{x \to 0^+} \frac{x^2 - 2}{x^4} = -\infty
\]

By symmetry,

\[
\lim_{x \to 0^-} \frac{x^2 - 2}{x^4} = -\infty.
\]

Hence we have a vertical asymptote at \( x = 0 \).

(iii) For large values of \( x \),

\[
\lim_{x \to \infty} \frac{x^2 - 2}{x^4} = 0.
\]

By symmetry,

\[
\lim_{x \to -\infty} \frac{x^2 - 2}{x^4} = 0.
\]

(iv) Now we consider the derivatives.

\[
\frac{dy}{dx} = \frac{2x^5 - 4x^3(x^2 - 2)}{x^8}
\]

\[= 2 \left( \frac{x^2 - 2(x^2 - 2)}{x^5} \right)
\]

\[= 2 \left( \frac{4 - x^2}{x^5} \right).
\]

Hence \( dy/dx = 0 \) when \( x = \pm2 \). Since \( dy/dx \) is defined for all \( x \) in the domain of our function, these are the only critical points.

\[
\frac{d^2y}{dx^2} = 2 \left( \frac{-2x^6 - 5x^4(4 - x^2)}{x^{10}} \right)
\]

\[= 2 \left( \frac{-2x^2 - 5(4 - x^2)}{x^6} \right)
\]

\[= 2 \left( \frac{3x^2 - 20}{x^6} \right).
\]
Since $x^6 > 0$ for all $x$ in the domain, the sign of $d^2y/dx^2$ depends solely on the sign of $3x^2 - 20$. Thus:

\[
\frac{d^2y}{dx^2} \begin{cases} 
= 0, & \text{if } x = \pm 2\sqrt{5/3}; \\
< 0, & \text{if } |x| < 2\sqrt{5/3}; \\
> 0, & \text{otherwise}.
\end{cases}
\]

Hence, since $\sqrt{5/3} > 1$, we see that $d^2y/dx^2 < 0$ when $x = \pm 2$. Hence we have a local maximum at $x = \pm 2$. Since the sign of $d^2y/dx^2$ changes at $x = \pm 2\sqrt{5/3}$ we see that the graph has a point of inflection at these points.

(v) This is enough information with which to sketch the graph.