MATH1003

ASSIGNMENT 7
ANSWERS

1. (i) Taking logs gives:

$$\begin{aligned}\ln y &= \ln\left((x+2)^{10}(2x-3)^4\right)\\ &= 10\ln(x+2) + 4\ln(2x-3).\end{aligned}$$

Differentiating, we obtain:

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{10}{x+2} + \frac{4}{2x-3}\\ \Rightarrow \frac{dy}{dx} &= y\left(\frac{10}{x+2} + \frac{4}{2x-3}\right)\\ &= (x+2)^{10}(2x-3)^4\left(\frac{10}{x+2} + \frac{4}{2x-3}\right)\\ &= 10(x+2)^9(2x-3)^4 + 4(x+2)^{10}(2x-3)^3.\end{aligned}$$

(ii) We begin by taking logs:

$$\begin{aligned}\ln y &= \ln\frac{(x+1)^4}{\sqrt{x^2-1}}\\ &= 4\ln(x+1) - \frac{1}{2}\ln(x^2-1).\end{aligned}$$

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Now we differentiate and simplify:

$$\begin{aligned}
\frac{1}{y}\frac{dy}{dx} &= \frac{4}{x+1} - \frac{1}{2} \times \frac{2x}{x^2-1} \\
\Rightarrow \frac{dy}{dx} &= y\left(\frac{4}{x+1} - \frac{x}{x^2-1}\right) \\
&= \frac{(x+1)^4}{\sqrt{x^2-1}}\left(\frac{4}{x+1} - \frac{x}{x^2-1}\right) \\
&= \frac{(x+1)^4}{\sqrt{x^2-1}} \times \frac{4(x^2-1) - x(x+1)}{(x+1)(x^2-1)} \\
&= \frac{(x+1)^3(4x^2-4-x^2-x)}{(x^2-1)^{3/2}} \\
&= \frac{(x+1)^3(3x^2-x-4)}{(x^2-1)^{3/2}} \\
&= \frac{(x+1)^3(x+1)(3x-4)}{(x^2-1)^{3/2}} \\
&= \frac{(3x-4)(x+1)^4}{(x^2-1)^{3/2}}.
\end{aligned}$$

2. (i) Recall from the Chain Rule that:

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}.$$

Hence

$$\frac{dy}{dx} = 3\sinh(3x)e^{\cosh 3x}.$$

(ii) We shall apply the Chain Rule. Let $u = \cosh x$. Then $y = \sinh u$ and:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\
&= \cosh u \times \sinh x \\
&= \sinh(x)\cosh(\cosh x).
\end{aligned}$$

(iii) First we shall calculate the derivative of $\sinh^{-1} 2x$. Let $u = \sinh^{-1} 2x$. Then $\sinh u = 2x$ and implicit differentiation gives:

$$\begin{aligned}
\frac{du}{dx}\cosh u &= 2 \\
\Rightarrow \qquad \frac{du}{dx} &= \frac{2}{\cosh u}.
\end{aligned}$$

Recalling that $\cosh^2 u - \sinh^2 u = 1$ we see that:

$$\cosh u = \sqrt{1 + \sinh^2 u} \\ = \sqrt{1 + 4x^2}.$$

Hence:

$$\frac{du}{dx} = \frac{2}{\sqrt{1+4x^2}}.$$

Now we apply the product rule to find dy/dx:

$$\begin{aligned} \frac{dy}{dx} &= 2x \sinh^{-1} 2x + x^2 \frac{d}{dx} \sinh^{-1} 2x \\ &= 2x \sinh^{-1} 2x + \frac{2x^2}{\sqrt{1+4x^2}} \\ &= 2x \left(\sinh^{-1} 2x + \frac{x}{\sqrt{1+4x^2}} \right). \end{aligned}$$

(iv) Let $u = \sinh x$, so that $y = \ln u$. By the Chain Rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{u} \times \cosh x \\ &= \frac{1}{\sinh x} \cosh x \\ &= \coth x. \end{aligned}$$

3. (i) We use implicit differentiation:

$$\begin{aligned} & x^2 - y^2 = 1 \\ \Rightarrow \quad & 2x - 2y\frac{dy}{dx} = 0 \\ \Rightarrow \quad & \frac{dy}{dx} = \frac{x}{y}. \end{aligned}$$

The tangent is parallel to the x-axis when:

$$\begin{aligned} & \frac{dy}{dx} = 0 \\ \Rightarrow & \frac{x}{y} = 0 \\ \Rightarrow & x = 0. \end{aligned}$$

When $x = 0$ we have:

$$\begin{aligned} & 0^2 - y^2 = 1 \\ \Rightarrow & y^2 = -1. \end{aligned}$$

This is impossible.

(ii) Using implicit differentiation we obtain:

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 2(1+xy)\left(y + x\frac{dy}{dx}\right) \\ \Rightarrow x + y\frac{dy}{dx} &= y + x\frac{dy}{dx} + xy^2 + x^2y\frac{dy}{dx} \\ \Rightarrow (y - x - x^2y)\frac{dy}{dx} &= y - x + xy^2 \\ \Rightarrow \frac{dy}{dx} &= \frac{y - x + xy^2}{y - x - x^2y}. \end{aligned}$$

This is zero when the numerator is zero. Hence:

$$\begin{aligned} y - x + xy^2 &= 0 \\ \Rightarrow x(1-y^2) &= y \\ \Rightarrow x &= \frac{y}{1-y^2}. \end{aligned}$$

We shall show that this is impossible. Substituting back into the equation for the curve we obtain:

$$\begin{aligned} \frac{y^2}{(1-y^2)^2} + y^2 &= \left(1 + \frac{y^2}{1-y^2}\right)^2 \\ &= \left(\frac{1-y^2+y^2}{1-y^2}\right)^2 \\ &= \frac{1}{(1-y^2)^2}, \\ \Rightarrow y^2 + y^2(1-y^2)^2 = 1 &\qquad \text{multiplying through by } (1-y^2)^2, \\ \Rightarrow y^6 - 2y^4 + 2y^2 - 1 = 0 &\qquad \text{expanding the brackets,} \\ \Rightarrow z^3 - 2z^2 + 2z - 1 = 0 &\qquad \text{setting } z = y^2, \\ \Rightarrow (z-1)(z^2 - z + 1) = 0. & \end{aligned}$$

Note that $z^2 - z + 1$ does not factorise. We see that the only solution is when $z = 1$; i.e. when $y^2 = 1$. But when this is the case x is undefined (since $x = \frac{y}{1-y^2}$). Hence no tangent line parallel to the x-axis can exist.

4. (i) Let $f(x) = 1/(5x-1)$. Then:

$$\begin{aligned} f'(x) &= \frac{-1 \cdot 5}{(5x-1)^2} \\ &= \frac{-5}{(5x-1)^2}, \\ f''(x) &= \frac{-5 \cdot -2 \cdot 5}{(5x-1)^3} \\ &= \frac{2 \cdot 5^2}{(5x-1)^3}. \end{aligned}$$

We see that:

$$f^{(n)}(x) = \frac{(-1)^n 5^n n!}{(5x-1)^{n+1}}.$$

(ii) Consider $h(\theta) = \theta e^{-\theta}$. Then:

$$\begin{aligned} h'(\theta) &= 1 \cdot e^{-\theta} + \theta \cdot -e^{-\theta} \\ &= e^{-\theta} - \theta e^{-\theta} \\ &= e^{-\theta} - h(\theta), \\ h''(\theta) &= -e^{-\theta} - h'(\theta) \\ &= -2e^{-\theta} + h(\theta), \\ h'''(\theta) &= 2e^{-\theta} + h'(\theta) \\ &= 3e^{-\theta} - h(\theta). \end{aligned}$$

Hence:

$$h^{(n)}(\theta) = (-1)^{n+1} n e^{-\theta} + (-1)^n h(\theta).$$

Thus:

$$h^{(n)}(0) = (-1)^{n+1} n.$$