1. Let \( g(x) + x \sin g(x) = x^2 \), and \( g(1) = 0 \). Differentiating both sides we obtain:

\[
g'(x) + \sin g(x) + xg'(x) \cos g(x) = 2x.
\]

Setting \( x = 1 \) gives:

\[
g'(1) + \sin 0 + g'(1) = 2.
\]

Rearranging we see that \( g'(1) = 1 \). Differentiating a second time gives:

\[
g''(x) + g'(x) \cos g(x) + g'(x) \cos g(x) + xg''(x) \cos g(x) - x(g'(x))^2 \sin g(x) = 2.
\]

We see that:

\[
g''(1) + 1 + 1 + g''(1) - 0 = 2
\]

and so \( g''(1) = 0 \).

2. (i) Differentiating both sides implicitly gives:

\[
\frac{dy}{dx} \sin x^2 + 2xy \cos x^2 = \sin y^2 + 2xy \frac{dy}{dx} \cos y^2
\]

\[
\Rightarrow \frac{dy}{dx} \sin x^2 - 2xy \frac{dy}{dx} \cos y^2 = \sin y^2 - 2xy \cos x^2
\]

\[
\Rightarrow (\sin x^2 - 2xy \cos y^2) \frac{dy}{dx} = \sin y^2 - 2xy \cos x^2
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{\sin y^2 - 2xy \cos x^2}{\sin x^2 - 2xy \cos y^2}
\]

(ii) Recall that:

\[
\frac{d}{dz} \cot z = -\csc^2 z.
\]

http://erdos.math.unb.ca/~kasprzyk/

kasprzyk@unb.ca.
Hence, via implicit differentiation of both sides:

\[ y + x \frac{dy}{dx} = -\left( y + x \frac{dy}{dx} \right) \csc^2(xy) \]

\[ \Rightarrow \quad x \frac{dy}{dx} + x \frac{dy}{dx} \csc^2(xy) = -y \csc^2(xy) - y \]

\[ \Rightarrow \quad x \left( 1 + \csc^2(xy) \right) \frac{dy}{dx} = -y \left( 1 + \csc^2(xy) \right) \]

\[ \Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}. \]

(iii) First we consider the right-hand side. Let \( u = xy^2 \). Then \( \sin(xy^2) = \sin u \), and by the Chain Rule:

\[ \frac{d}{dx} \sin(xy^2) = \frac{d}{du} \sin u \times \frac{du}{dx} \]

\[ = \cos u \times \left( y^2 + 2xy \frac{dy}{dx} \right) \]

\[ = \cos(xy^2) \left( y^2 + 2xy \frac{dy}{dx} \right). \]

Hence:

\[ 1 + x = \sin(xy^2) \]

\[ \Rightarrow \quad 1 = \cos(xy^2) \left( y^2 + 2xy \frac{dy}{dx} \right) \]

\[ \Rightarrow \quad 2xy \cos(xy^2) \frac{dy}{dx} = 1 - y^2 \cos(xy^2) \]

\[ \Rightarrow \quad \frac{dy}{dx} = \frac{1 - y^2 \cos(xy^2)}{2xy \cos(xy^2)} \]

\[ = \frac{1}{2xy} \sec(xy^2) - \frac{y}{2x}. \]

3. (i) Let \( y = \tan^{-1} x \). Then \( \tan y = x \). Differentiating both sides we obtain:

\[ \sec^2 y \frac{dy}{dx} = 1 \]

\[ \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} \]

\[ = \frac{1}{1 + \tan^2 y} \]

\[ = \frac{1}{1 + x^2}. \]
(ii) Let $u = \sqrt{x}$, so that $y = \tan^{-1} u$. By the Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$ 

Hence, using our answer to (i), we obtain:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \times \frac{1}{1 + u^2}$$

$$= \frac{1}{2\sqrt{x}} \times \frac{1}{1 + (\sqrt{x})^2}$$

$$= \frac{1}{2\sqrt{x}(1 + x)}.$$ 

(iii) First we calculate the derivative of $\cos^{-1} x$. Let $u = \cos^{-1} x$. Then $\cos u = x$.

Differentiating gives:

$$-\sin u \frac{du}{dx} = 1$$

$$\Rightarrow \quad \frac{du}{dx} = -\frac{1}{\sin u}$$

$$= -\frac{1}{\sqrt{1 - \cos^2 u}}$$

$$= -\frac{1}{\sqrt{1 - x^2}}.$$ 

Using our answer to (i) we see that:

$$\frac{dy}{dx} = \frac{1}{1 + x^2} - \frac{1}{\sqrt{1 - x^2}}.$$ 

4. From the graph of $x^2 - xy + y^2 = 3$ we see that the minimum and maximum values of $y$ occur when the tangent is parallel to the $x$-axis. The minimum and maximum values of $x$ occur when the tangent is parallel to the $y$-axis. Differentiating implicitly we find that:

$$2x - y - x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$$

$$\Rightarrow (2y - x) \frac{dy}{dx} = y - 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - 2x}{2y - x}.$$
The tangent is parallel to the $x$-axis when $y - 2x = 0$, i.e. when $y = 2x$. Substituting this back into the equation of the tilted ellipse gives:

$$x^2 - x(2x) + (2x)^2 = 3$$
\[\Rightarrow x^2 - 2x^2 + 4x^2 = 3\]
\[\Rightarrow x^2 = 1\]
\[\Rightarrow x = \pm 1.\]

Thus the minimum value of $y$ is $-2$ and the maximum value of $y$ is $2$.

The tangent is parallel to the $y$-axis when $2y - x = 0$, i.e. when $x = 2y$. Substituting this back into the equation gives:

$$(2y)^2 - (2y)y + y^2 = 3$$
\[\Rightarrow y = \pm 1.\]

We see that the minimum value of $x$ is $-2$ and the maximum value of $x$ is $2$.

5. Differentiating $(x^2 + y^2)^2 = 2(x^2 - y^2)$ gives:

$$2(x^2 + y^2) \left(2x + 2y \frac{dy}{dx}\right) = 4 \left(x - y \frac{dy}{dx}\right).$$

Setting $\frac{dy}{dx} = 0$, we obtain:

$$4x(x^2 + y^2) = 4x.$$ 

Hence either $x = 0$ or $x^2 + y^2 = 1$. Let us consider the second possibility.

Substituting into the equation of the lemniscate we obtain:

$$1^2 = 2(x^2 - (1 - x^2))$$
\[\Rightarrow \frac{1}{2} = 2x^2 - 1\]
\[\Rightarrow x^2 = \frac{3}{4}\]
\[\Rightarrow x = \pm \frac{\sqrt{3}}{2}.\]

When $x^2 = \frac{3}{4}$ we see that $y^2 = 1 - \frac{3}{4} = \frac{1}{4}$.

We have found that $\frac{dy}{dx} = 0$ at five points: when $(x, y)$ is equal to $(0,0), \ (\frac{-\sqrt{3}}{2}, \frac{-1}{2}), \ (\frac{\sqrt{3}}{2}, \frac{-1}{2}), \ (\frac{-\sqrt{3}}{2}, \frac{1}{2}), \ (\frac{\sqrt{3}}{2}, \frac{1}{2})$. 
Consulting the graph we see that we have a slight problem. There should be only four points where the tangent line is parallel to the $x$-axis. Somehow the extra point $(0,0)$ has appeared.

This is because the graph is rather unusual at $(0,0)$. Close to the origin the graph looks like an $\times$. It is what we call a *singularity*. It is because of this singularity that we are getting our extra point; we should simply ignore the solution $(0,0)$ as not relevant to the answer.