MATH1003

ASSIGNMENT 2
ANSWERS

1. (i) By definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Hence:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\frac{3+x+h}{1-3(x+h)} - \frac{3+x}{1-3x}}{h} \\
&= \lim_{h \to 0} \frac{(3+x+h)(1-3x) - (3+x)(1-3x-3h)}{h(1-3x)(1-3x-3h)} \\
&= \lim_{h \to 0} \frac{10}{(1-3x)(1-3x-3h)} \\
&= \frac{10}{(1-3x)^2}.
\end{aligned}$$

(ii)

$$\begin{aligned}
g'(x) &= \lim_{h \to 0} \frac{(x+h+\sqrt{x+h}) - (x+\sqrt{x})}{h} \\
&= \lim_{h \to 0} \frac{h + \sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \frac{h}{h} + \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= 1 + \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\
&= 1 + \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= 1 + \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= 1 + \frac{1}{2\sqrt{x}}.
\end{aligned}$$

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(iii)

$$
\begin{aligned}
C'(x) &= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x^2}{(x+h)^2x^2} - \frac{(x+h)^2}{x^2(x+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{-2xh - h^2}{hx^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{-2x - h}{x^2(x+h)^2} \\
&= \frac{-2x}{x^2x^2} \\
&= -\frac{2}{x^3}.
\end{aligned}
$$

2. Let $f(x) = x^2 + x$. Then $f'(x) = 2x + 1$. Hence the tangent to the point $(a, f(a))$ has equation:

$$
\begin{aligned}
y - f(a) &= (2a+1)(x-a), \\
\Rightarrow \quad y - a^2 - a &= (2a+1)x - 2a^2 - a, \\
\Rightarrow \quad y &= (2a+1)x - a^2.
\end{aligned}
$$

Setting $x = 2$, $y = -3$, we shall solve for a:

$$
\begin{aligned}
-3 &= 2(2a+1) - a^2, \\
\Rightarrow \quad 0 &= a^2 - 4a - 5, \\
\Rightarrow \quad 0 &= (a+1)(a-5), \\
\Rightarrow \quad a &= -1 \text{ or } 5.
\end{aligned}
$$

Hence we see that the two tangents passing through $(2, -3)$ have equations:

$$
\begin{aligned}
y &= -(x+1), && \text{when } a = -1; \\
y &= 11x - 25, && \text{when } a = 5.
\end{aligned}
$$

3. (i)

$$
\begin{aligned}
f(x) &= \int (2x-3)dx \\
&= \int 2x\,dx - \int 3\,dx \\
&= 2\int x\,dx - 3\int dx \\
&= 2\times\frac{x^2}{2} - 3\times x + c, \quad \text{where } c \text{ is an arbitrary constant} \\
&= x^2 - 3x + c.
\end{aligned}
$$

(ii)

$$
\begin{aligned}
P(x) &= \int \left(\frac{5}{x^6} + 7x^6\right) dx \\
&= 5\int x^{-6}\,dx + 7\int x^6\,dx \\
&= 5\times\frac{x^{-5}}{-5} + 7\times\frac{x^7}{7} + c, \quad \text{where } c \text{ is an arbitrary constant} \\
&= x^7 - \frac{1}{x^5} + c.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
Q(t) &= \int \frac{t^{31}-1}{t^2} dt \\
&= \int t^{29}\,dt - \int t^{-2}\,dt \\
&= \frac{1}{30}t^{30} - \frac{1}{-1}t^{-1} + c, \quad \text{where } c \text{ is an arbitrary constant} \\
&= \frac{1}{30}t^{30} + \frac{1}{t} + c.
\end{aligned}
$$

4.

$$
\begin{aligned}
\int_0^a (3x^2 - 18x + 14)dx &= \left[3\times\frac{x^3}{3} - 18\times\frac{x^2}{2} + 14\times x\right]_0^a \\
&= \left[x^3 - 9x^2 + 14x\right]_0^a \\
&= a^3 - 9a^2 + 14a \\
&= a(a^2 - 9a + 14) \\
&= a(a-2)(a-7)
\end{aligned}
$$

Hence the integral is zero when a equals $0, 2$, or 7.

5. Let:

$$g(x) = \frac{5 - 4x^3 + 2x^6}{x^6}.$$

Rewriting this fraction gives $g(x) = 5x^{-6} - 4x^{-3} + 2$. Hence an antiderivative is given by:

$$\begin{aligned} G(x) &= \frac{5}{-5}x^{-5} - \frac{4}{-2}x^{-2} + 2x + c \\ &= -\frac{1}{x^5} + \frac{2}{x^2} + 2x + c, \end{aligned}$$

where c is an arbitrary constant.