M3P23, M4P23, M5P23: COMPUTATIONAL ALGEBRA & GEOMETRY
EXAM 2014

(1) (i) State clearly the definition of a *Gröbner basis*. State Buchberger's Criterion for determining when a generating set $G = \{g_1, \ldots, g_m\}$ for an ideal $I \subset k[x_1, \ldots, x_n]$ is a Gröbner basis.

(ii) Using lex order, for which values of a and b is the set $\{y - ax^2, z - by^3\} \subset \mathbb{C}[x, y, z]$ a Gröbner basis?

(iii) State the definition of a *reduced* Gröbner basis. For every choice of a and b write down a lex-ordered reduced Gröbner basis for the set in (ii).

(iv) Hence or otherwise, for which values of a_1, b_1, and a_2, b_2 are the ideals $(y-a_1x^2, z-b_1y^3)$ and $(y - a_2x^2, z - b_2y^3)$ equal?

(2) (i) Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be an ideal. Recall that the *l-th elimination ideal* I_l is defined by $I_l := I \cap \mathbb{C}[x_{l+1}, \ldots, x_n]$.

(a) Prove that I_l is an ideal in $\mathbb{C}[x_{l+1}, \ldots, x_n]$.

(b) Prove that I_{l+1} is the first elimination ideal of I_l.

(ii) Let $I = (f_1, \ldots, f_s) \subset \mathbb{C}[x_1, \ldots, x_n]$ and assume that, for some $1 \leq i \leq s$, f_i can be written in the form

$$f_i = cx_1^N + \text{terms in which } x_1 \text{ has degree} < N,$$

where $c \in \mathbb{C}\backslash\{0\}$ and $N > 0$. Prove that given a partial solution $(a_2, \ldots, a_n) \in \mathbb{V}(I_1)$ there exists some $a_1 \in \mathbb{C}$ such that $(a_1, a_2, \ldots, a_n) \in \mathbb{V}(I)$.
[Hint: You may wish to make use of the Extension Theorem, which you should quote.]

(iii) Find all solutions in $\mathbb{C}^3$ to the system of equations:

$$x^2 + \frac{1}{x^2} = y, \qquad x + \frac{1}{x} = z.$$

You may assume that a lex-ordered Gröbner basis for the corresponding ideal is $\{x^2 - xz + 1, y - z^2 + 2\}$. What are the irreducible components of the resulting affine variety?

(3) (i) Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. State clearly the definition of the *affine variety* $\mathbb{V}(I) \subset k^n$. Show that if I_1, I_2 are two ideals such that $I_1 \subseteq I_2$, then $\mathbb{V}(I_1) \supseteq \mathbb{V}(I_2)$.

(ii) The variety $\mathbb{V}(y - x^2, z - x^3) \subset \mathbb{R}^3$ is called the *twisted cubic*. Prove that $\mathbb{V}((y - x^2)^2 + (z - x^3)^2)$ is also the twisted cubic.

(iii) Let $I \subset \mathbb{R}[x_1, \ldots, x_n]$ be an ideal. Show that any variety $\mathbb{V}(I) \subset \mathbb{R}^n$ can be defined by a single equation.

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(iv) If we replace $\mathbb{R}$ with $\mathbb{C}$ in (iii) is the claim still true? Justify your answer with either a proof or a counter-example.

(4) (i) Let $I \subset k[x_1, \dots, x_n]$ be an ideal. State the definition of the *radical* $\sqrt{I}$ of I. State a version of the Nullstellensatz that connects radical ideals with affine varieties.

(ii) Consider the ideal $I = \big(x^2 - x - 2, x(y^2 - 1)\big) \subset \mathbb{C}[x, y]$. By computing $\mathbb{V}(I) \subset \mathbb{C}^2$, write down $\sqrt{I}$. Hence or otherwise show that $y^2 - 1 \in \sqrt{I}$.

(iii) Let $I_i \subset k[x_1, \dots, x_n]$ be an ideal for each i in some finite indexing set Γ. Prove that

$$\sqrt{\bigcap_{i \in \Gamma} I_i} = \bigcap_{i \in \Gamma} \sqrt{I_i}.$$

(iv) By writing $f \in \mathbb{C}[x]$, $f \neq 0$, as a product of distinct linear factors

$$f = c \prod_{i=1}^{d} (x - a_i)^{r_i},$$

where $c, a_i \in \mathbb{C}$, $c \neq 0$, use the result in (iii) to show that

$$\sqrt{(f)} = \left(\prod_{i=1}^{d} (x - a_i) \right).$$

(5) **Master Question.** Let $I \subset \mathbb{C}[x_1, \dots, x_n]$ be an ideal, and fix a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$. We define the *saturation* of I with respect to f to be

$$I : (f^\infty) := \{g \in \mathbb{C}[x_1, \dots, x_n] \mid f^m g \in I \text{ for some } m > 0\}.$$

(i) Prove that $I : (f^\infty)$ is an ideal.

(ii) State the definition of *colon ideal* and prove that we have an ascending chain of ideals

$$I : (f) \subseteq I : (f^2) \subseteq I : (f^3) \subseteq \dots$$

Hence or otherwise show that there exists some $N \in \mathbb{Z}_{>0}$ such that $I : (f^\infty) = I : (f^N)$.

(iii) Given $I = (f_1, \dots, f_s) \subset \mathbb{C}[x_1, \dots, x_n]$ and $f \in \mathbb{C}[x_1, \dots, x_n]$, define

$$\tilde{I} := (f_1, \dots, f_s, 1 - fy) \subset \mathbb{C}[x_1, \dots, x_n, y],$$

where y is a new variable. Prove that $I : (f^\infty) = \tilde{I} \cap \mathbb{C}[x_1, \dots, x_n]$.