M3P23, M4P23, M5P23: COMPUTATIONAL ALGEBRA & GEOMETRY
EXAM 2013

(1) (i) State clearly the definition of a *Gröbner basis*. State Buchberger's Criterion for determining when a generating set $G = \{g_1, \ldots, g_m\}$ for an ideal $I \subset k[x_1, \ldots, x_n]$ is a Gröbner basis.

(ii) Find a Gröbner basis for the ideal $I = (y^2 - x, z^4 - y^2) \subset \mathbb{C}[x, y, z]$ using lex order.

(iii) Recall that a Gröbner basis G is said to be *reduced* if, for all $g \in G$, $\mathrm{LC}(g) = 1$ and no monomial of g lies in $(\mathrm{LT}(G \setminus \{g\}))$. Is the basis you found in (ii) reduced? If not, make it reduced.

(iv) A computer program outputs $\{x - y^2, y^3 - z^4\}$ as a Gröbner basis for the ideal I defined in (ii). What conclusions can you draw?

(2) (i) Assuming the Ascending Chain Condition for ideals, prove that for any descending chain of affine varieties

$$V_1 \supseteq V_2 \supseteq \ldots$$

there exists some positive integer N such that $V_N = V_{N+1} = \ldots$.

(ii) Recall that an affine variety $V \subset k^n$ is said to be *irreducible* if whenever $V = U \cup W$ for two affine varieties $U, W \subset k^n$, then either $V = U$ or $V = W$. By using the result of (i), show that any affine variety V can be written as a finite union of irreducible varieties:

$$V = V_1 \cup \ldots \cup V_m.$$

(iii) Show that if $g \in k[x_1, \ldots, x_n]$ factors as $g = g_1 g_2$ then, for any $f \in k[x_1, \ldots, x_n]$ we have that $\mathbb{V}(f, g) = \mathbb{V}(f, g_1) \cup \mathbb{V}(f, g_2)$.

(iv) Write $\mathbb{V}\left(y^2 - x^2, x(z - y) + z(z - y)\right) \subset \mathbb{C}^3$ as a finite union of irreducible varieties. [Hint: You may assume that any variety $V \subset \mathbb{C}^3$ that can be defined parametrically is irreducible.]

(3) (i) Let $f \in \mathbb{C}[x]$ be a non-zero polynomial. We can express f as a product of linear factors

$$f = c \prod_{i=1}^{d} (x - a_i)^{r_i},$$

where $c, a_i \in \mathbb{C}$, $c \neq 0$. Define the *reduction* of f to be

$$f_{red} = c \prod_{i=1}^{d} (x - a_i) \in \mathbb{C}[x].$$

Compute $\mathbb{V}(f)$ and show directly that $\mathbb{I}(\mathbb{V}(f)) = (f_{red}) \subset \mathbb{C}[x]$.

a.m.kasprzyk@imperial.ac.uk
http://magma.maths.usyd.edu.au/~kasprzyk/.

(ii) Define the *radical* $\sqrt{I}$ of a polynomial ideal I and state a version of the Nullstellensatz relating $\mathbb{I}(\mathbb{V}(I))$ and $\sqrt{I}$.

(iii) Let $I = (xy, x(x-y)) \subset \mathbb{C}[x,y]$. Describe $\mathbb{V}(I)$ and find $\sqrt{I}$.

(4) (i) Prove that if

$$I = (x^\alpha \mid \alpha \in A), \qquad A \subset \mathbb{Z}^n_{\geq 0}$$

is a monomial ideal, then a monomial x^β is an element of I if and only if x^β is divisible by some $\alpha \in A$.

(ii) Recall that, given an ideal $I \subset k[x_1, \ldots, x_n]$, we define the *leading term ideal* of I to be the ideal generated by the leading terms of the elements in I, i.e.

$$\mathrm{LT}(I) = (\mathrm{LT}(f) \mid f \in I).$$

Let $I = \left(x^3 - 2xy, x^2y - 2y^2 + x\right) \subset k[x,y]$. Using graded lex order, show that $x^2 \in \mathrm{LT}(I)$. Is it the case that when $I = (f_1, \ldots, f_m)$ we have $\mathrm{LT}(I) = (\mathrm{LT}(f_1), \ldots, \mathrm{LT}(f_m))$?

(iii) Suppose that $I = (f_1, \ldots, f_m)$ is a polynomial ideal such that

$$(\mathrm{LT}(f_1), \ldots, \mathrm{LT}(f_m)) \subsetneqq \mathrm{LT}(I).$$

Prove that there exists some $f \in I$ whose remainder on division by $f_1, \ldots, f_m$ is non-zero.

(5) **Mastery Question.** Fix a monomial order and let $G = \{g_1, \ldots, g_m\} \subset k[x_1, \ldots, x_n]$ be a set of polynomials. Given $f \in k[x_1, \ldots, x_n]$, we say that f *reduces to zero modulo* G, and write $f \rightarrow_G 0$, if f can be expressed in the form

$$f = a_1 g_1 + \ldots + a_m g_m,$$

such that whenever $a_i g_i \neq 0$ we have that

$$\mathrm{multideg}\,(f) \geq \mathrm{multideg}\,(a_i g_i).$$

(i) By using the Division Algorithm, prove that $\overline{f}^G = 0$ implies that $f \rightarrow_G 0$.

(ii) Let $f = x(y+1)(y-1)$ and $G = \{xy + 1, y^2 - 1\}$. Using lex order, show that the converse to (i) does not hold.

(iii) Let $f, g \in G$ be such that the leading monomials of f and g are coprime, i.e.

$$\mathrm{lcm}\{\mathrm{LM}(f), \mathrm{LM}(g)\} = \mathrm{LM}(f)\,\mathrm{LM}(g).$$

Prove that $S(f,g) \rightarrow_G 0$.
[Hint: Without loss of generality you may assume that $\mathrm{LC}(f) = \mathrm{LC}(g) = 1$. It may also be helpful to write $f = \mathrm{LM}(f) + p$ and $g = \mathrm{LM}(g) + q$ for some polynomials $p, q \in k[x_1, \ldots, x_n]$.]

Recall that Buchberger's Criterion states that a set of generators $G = \{g_1, \ldots, g_m\}$ for an ideal $I \subset k[x_1, \ldots, x_n]$ is a Gröbner basis if and only if the remainder $\overline{S(g_i, g_j)}^G$

is zero for all $i \neq j$. The proof relies on the fact that we can express S-polynomials in the form

$$S(g_i, g_j) = \sum_{r=1}^{m} a_{rij} g_r, \qquad \text{where multideg}\,(a_{rij} g_r) \leq \text{multideg}\,(S(g_i, g_j))\,.$$

More generally, we can restate Buchberger's Criterion as follows: $G = \{g_1, \ldots, g_m\}$ is a Gröbner basis for a polynomial ideal I if and only if $S(g_i, g_j) \to_G 0$ for all $i \neq j$.

(iv) Explain briefly how your result in (iii) can be used to simplify the calculations involved when applying Buchberger's Criterion.

(v) Using graded lex order, determine whether $G = \{x^3 + y, y(1 + z), z^4\}$ is a Gröbner basis for the polynomial ideal $I = (x^3 + y, y(1 + z), z^4)$.