M3P23, M4P23, M5P23: COMPUTATIONAL ALGEBRA & GEOMETRY

SOLUTIONS 5

(1) (a) Let $I = \big(x^3 - y^2, 3x^2, 2y\big) \subset k^2$. This has lex Gröbner basis

$$G = \{x^2, y\}.$$

Hence $\mathbb{V}(I) = \{(0,0)\}$ and so $(0,0)$ is the only singular point of $x^3 = y^2$.

(b) Let $I = \big(x^3 - cx^2 + y^2, 3x^2 - 2cx, 2y\big) \subset k^2$. To calculate the lex Gröbner basis of this using a computer, I'm going to regard c as a variable and use the order $c > x > y$. Using MAGMA:

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> R<c,x,y>:=PolynomialRing(Rationals(),3);
> I:=ideal<R|[x^3-c*x^2+y^2,3*x^2-2*c*x,2*y]>;
> GroebnerBasis(I);
[
    c*x - 3/2*x^2,
    x^3,
    y
]
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Again, we see that $(0,0)$ is the only singular point. (Of course we can solve this without using a computer – or even calculating a Gröbner basis. See the solution I offer for the next question.)

(c) Since we're told that $x^2+y^2 = a^2$ is a circle, we have that $a \neq 0$. Let $I = (x^2+y^2 - a^2, 2x, 2y)$. Notice that $x \in I$ and $y \in I$, hence $\mathbb{V}(I) \subset \mathbb{V}(x) \cap \mathbb{V}(y) = \{(0,0)\}$. But $(0,0)$ is not a solution to $x^2 + y^2 = a^2$ except in the case $a = 0$ (which has already been excluded). Hence there are no singular points.

(2) (a) Let $I_1 = (y - x^2, z - x^3)$ and $I_2 = ((y-x^2)^2 + (z-x^3)^2)$. Since $I_2 \subset I_1$, we have that $\mathbb{V}(I_1) \subset \mathbb{V}(I_2)$. Conversely let $(a,b,c) \in \mathbb{V}(I_2) \subset \mathbb{R}^3$, so that $(b-a^2)^2 + (c-a^3)^2 = 0$. The only possibility is that both $b - a^2 = 0$ and $c - a^3 = 0$ (since we're working over $\mathbb{R}$), hence $(a,b,c) \in \mathbb{V}(I_1)$. Hence $\mathbb{V}(I_1) = \mathbb{V}(I_2)$.

(b) Let $I = (f_1, \ldots, f_m) \subset \mathbb{R}[x_1, \ldots, x_n]$ be an ideal, and set

$$g = f_1^2 + \ldots + f_m^2.$$

The ideal $I' = (g)$ is contained in I, and so $\mathbb{V}(I) \subset \mathbb{V}(I')$. Conversely let $(a_1, \ldots, a_n) \in \mathbb{V}(I')$, so that

$$f_1(a_1, \ldots, a_n)^2 + \ldots + f_m(a_1, \ldots, a_n)^2 = 0.$$

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Since we're working over $\mathbb{R}$, it much be that $f_i(a_1, \dots, a_n) = 0$ for each $1 \le i \le m$. Hence $(a_1, \dots, a_n) \in \mathbb{V}(I)$ and so $\mathbb{V}(I') \subset \mathbb{V}(I)$. We conclude that $\mathbb{V}(I) = \mathbb{V}(I')$, as required.

(3) Let $I = (x^2, y^2)$. Then $x, y \in \sqrt{I}$, and so $(x, y) \subset \sqrt{I}$. Suppose that there exists some $f \in \sqrt{I}$ such that $f \notin (x, y)$. We have that

$$f = \sum_{i \ge 0, j \ge 0} c_{ij} x^i y^j, \qquad \text{where } c_{ij} \in k,$$

where finitely many of the c_{ij} are non-zero. Rewriting this in the form

$$f = c_{00} + x \sum_{i>0, j \ge 0} c_{ij} x^{i-1} y^j + y \sum_{j>0} c_{0j} y^{j-1}$$

we see that $f \notin (x, y)$ implies that $c_{00} \neq 0$. In particular, $c_{00} \in \sqrt{I}$ and so $1 \in \sqrt{I}$. By definition of radical this implies that $1 \in I$, which is a contradiction.

Notice that the same argument works if we start with $I = (x^n, y^m)$ for any $n, m \in \mathbb{Z}_{>0}$.

(4) (a) Set $I = \big(x^3, y^3, xy(x+y)\big) \subset k[x, y]$ and consider $\tilde{I} = \big(x^3, y^3, xy(x+y), 1 - z(x+y)\big) \subset k[x, y, z]$. Using lex order, this has Gröbner basis $\{1\}$, so we conclude that $x + y \in \sqrt{I}$.

The lex-ordered Gröbner basis for I is $G = \{x^3, x^2y + xy^2, y^3\}$. By calculating $\overline{(x+y)^n}^G$ for successive powers of n, we find that $n = 3$ is the smallest power contained in I. [By looking at G, do you see how you could have deduced this without having to calculate the remainders?]

(b) We have $\tilde{I} = \big(x + z, x^2y, x - z^2, 1 - w(x^2 + 3xz)\big) \in k[x, y, z, w]$. Using lex order, this has Gröbner basis $G = \{x - 1, y, z + 1, w + 1/2\}$. This tells us that $1 \notin \tilde{I}$, and so $x^2 + 3xz \notin \sqrt{I}$.

(5) Let J be a prime ideal containing I. If $f \in \mathbb{C}[x_1, \dots, x_n]$ is such that $f^k \in I$ for some $k > 0$, then $f^k \in J$. But J is prime, so $f \in J$. Hence

$$\sqrt{I} \subseteq \bigcap J.$$

Conversely, suppose that $f \in \mathbb{C}[x_1, \dots, x_n]$ is such that $f \notin \sqrt{I}$. We shall show that the difference $\mathbb{V}(I) \setminus \mathbb{V}(f)$ is non-empty. Hence there exists some point $(a_1, \dots, a_n) \in \mathbb{V}(I) \setminus \mathbb{V}(f)$, and this corresponds (since $\mathbb{C}$ is algebraically closed) to a maximal ideal $(x_1 - a_1, \dots, x_n - a_n) \subset \mathbb{C}[x_1, \dots, x_n]$ containing I (since I is proper) but not containing f. Since maximal ideals are always prime, we conclude that $f \notin \bigcap J$.

It remains to show that $\mathbb{V}(I) \setminus \mathbb{V}(f) \neq \emptyset$. Suppose for a contradiction that $\mathbb{V}(I) \subseteq \mathbb{V}(f)$. By the Nullstellensatz we have that $\sqrt{(f)} \subseteq \sqrt{I}$, and so $f \in \sqrt{f}$. But this contradicts our choice of f.

[Those of you who have taken a course in commutative algebra should be able to provide a more direct proof of this result.]

(6) Let $I = (xz - y^2, z^3 - x^5) \subset \mathbb{C}[x, y, z]$. Using lex order, this has Gröbner basis

$$G = \{x^5 - z^3, x^4y^2 - z^4, x^3y^4 - z^5, x^2y^6 - z^6, xy^8 - z^7, xz - y^2, y^{10} - z^8\}.$$

From $y^{10} = z^8$ we see that

$$\mathbb{V}(I_1) = \big((\pm t^4, t^5) \mid t \in \mathbb{C}\big).$$

Extending we have that

$$\mathbb{V}(I) = \big((t^3, \pm t^4, t^5) \mid t \in \mathbb{C}\big).$$

(To see this I used the first equation, $x^5 = z^3$, and then checked that the parameterisations satisfied the remaining equations.) Thus we have expressed $\mathbb{V}(I)$ as the union of two components:

$$\mathbb{V}(I) = \big((t^3, t^4, t^5) \mid t \in \mathbb{C}\big) \cup \big((t^3, -t^4, t^5) \mid t \in \mathbb{C}\big).$$

The fact that these components are irreducible is an immediate consequence of them being defined parametrically.

(7) Let $V, W \subset k^n$ be two affine varieties such that $V \subset W$, and let Z be an irreducible component of V. Let $W = W_1 \cup \ldots \cup W_m$ be a decomposition of W into irreducible components. We will proceed by induction on the number of components m.

First suppose that $m = 1$. Since $Z \subset V \subset W = W_1$, we are done. Suppose now that $m > 1$. Since $Z \subset V \subset W$,

$$Z = \bigcup_{i=1}^{m} Z \cap W_i,$$

where each of the affine varieties $Z \cap W_i$ is disjoint. Without loss of generality we may assume that $Z \cap W_1 \neq \emptyset$. Set

$$Z_1 := Z \cap W_1, \qquad Z_2 := \bigcup_{i=2}^{m} Z \cap W_i.$$

Notice that Z_2, as the finite union of $(m-1)$ affine varieties, is also an affine variety. We have that $Z = Z_1 \cup Z_2$, and since Z is irreducible either $Z = Z_1$ or $Z = Z_2$. In the first case we conclude that $Z \subset W_1$, and so are done. In the second case we can apply our inductive hypothesis to the $(m-1)$ components $W_2 \cup \ldots \cup W_m$, and so are done.