M3P23, M4P23, M5P23: COMPUTATIONAL ALGEBRA & GEOMETRY

SOLUTIONS 3

(1) Let $S := \{\beta \mid x^\beta \in I\} \subset \mathbb{Z}^n_{\geq 0}$. SInce we are using a monomial order, S has a smallest element $\gamma \in S$. Then $x^\gamma \in I$, so there exists some $\alpha \in A$ such that $x^\alpha \mid x^\gamma$. Hence $\alpha \leq \gamma$. But $\alpha \in S$ by construction, so $\alpha = \gamma$.

(2) First we show existence. By Dickson's Lemma we can write $I = (x^{\alpha_1}, \ldots, x^{\alpha_s})$ for some finite set of generators $A = \{\alpha_1, \ldots, \alpha_s\} \subset \mathbb{Z}^n_{\geq 0}$. Suppose that there exist $\alpha_i, \alpha_j \in A$, $i \neq j$, such that $x^{\alpha_i} \mid x^{\alpha_j}$. Then $A' := A \setminus \{\alpha_j\}$ is such that $(x^\alpha \mid \alpha \in A') = I$. Proceeding by induction we see that this process must terminate (since A is finite) with a minimal generating set.

Now for uniqueness. Suppose for a contradiction that $\{x^{\alpha_1}, \ldots, x^{\alpha_s}\}$ and $\{x^{\beta_1}, \ldots, x^{\beta_r}\}$ are two different minimal generating sets. Without loss of generality we may take $x^{\beta_1} \notin \{x^{\alpha_1}, \ldots, x^{\alpha_s}\}$. Since $x^{\beta_1} \in (x^{\alpha_1}, \ldots, x^{\alpha_s})$, so there exists some α_i such that $x^{\alpha_i} \mid x^{\beta_1}$. But $x^{\alpha_i} \in (x^{\beta_1}, \ldots, x^{\beta_r})$, so there exists some β_j such that $x^{\beta_j} \mid x^{\alpha_i}$. Hence $x^{\beta_j} \mid x^{\beta_1}$. By minimality $j = 1$, hence $\alpha_i = \beta_1$.

(3) Suppose that $f \in I$. Then $f = \sum_{i=1}^s h_i x^{\alpha_i}$ for some $h_i \in k[x_1, \ldots, x_n]$. So each term of f is divisible by some x^{α_i}. Hence $\bar{f}^{x^{\alpha_1}, \ldots, x^{\alpha_s}} = 0$. Conversely suppose that $\bar{f}^{x^{\alpha_1}, \ldots, x^{\alpha_s}} = 0$. This means (by the Division Algorithm) that there exist $h_i \in k[x_1, \ldots, x_n]$ such that $f = \sum_{i=1}^s h_i x^{\alpha_i}$, and so $f \in I$.

(4) (a)

$$\begin{aligned}\frac{x^2yz^2}{4x^2z}(4x^2z - 7y^2) - \frac{x^2yz^2}{xyz^2}(xyz^2 + 3xz^4) &= x^2yz^2 - \frac{7}{4}y^3z - x^2yz^2 - 3x^2z^4 \\ &= -3x^2z^4 - \frac{7}{4}y^3z.\end{aligned}$$

(b)

$$\begin{aligned}\frac{x^4yz^2}{x^4y}(x^4y - z^2) - \frac{x^4yz^2}{3xyz^2}(3xyz^2 - y) &= x^4yz^2 - z^4 - x^4yz^2 + \frac{1}{3}x^3y \\ &= \frac{1}{3}x^3y - z^4.\end{aligned}$$

(c)

$$\begin{aligned}\frac{xyz^2}{xy}(xy + z^3) - \frac{xyz^2}{z^2}(z^2 - 3z) &= xyz^2 + z^3 - xyz^2 + 3xyz \\ &= 3xyz + z^3.\end{aligned}$$

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(5) We use Buchberger's Criterion.

$$\begin{aligned} S(x^4y^2 - z^5, x^3y^3 - 1) &= \frac{x^4y^3}{x^4y^2}(x^4y^2 - z^5) - \frac{x^4y^3}{x^3y^3}(x^3y^3 - 1) \\ &= x^4y^3 - yz^5 - x^4y^3 + x \\ &= -yz^5 + x. \end{aligned}$$

But $\overline{-yz^5 + x}^G = -yz^5 + x$, so this is not a Gröbner basis.

(6)

$$\begin{aligned} S(x^\alpha f, x^\beta g) &= \frac{x^\delta}{x^\alpha \mathrm{LT}(F)} x^\alpha f - \frac{x^\delta}{x^\beta \mathrm{LT}(g)} x^\beta g \\ &\qquad \text{where } \delta := \mathrm{lcm}\Big\{x^\alpha \mathrm{LM}(f)\,, x^\beta \mathrm{LM}(g)\Big\} \\ &= \frac{x^\delta}{\mathrm{LT}(F)} f - \frac{x^\delta}{\mathrm{LT}(g)} g \\ &= x^{\delta-\epsilon}\left(\frac{x^\epsilon}{\mathrm{LT}(f)} f - \frac{x^\epsilon}{\mathrm{LT}(g)} g\right) \\ &\qquad \text{where } \epsilon := \mathrm{lcm}\{\mathrm{LM}(f)\,, \mathrm{LM}(g)\} \\ &= x^{\delta-\epsilon} S(f, g). \end{aligned}$$

(7) (a) Let $I_1 = \mathbb{I}(V)$ and $I_2 = \mathbb{I}(W)$. Since V and W are both affine varieties, $\mathbb{V}(I_1) = V$ and $\mathbb{V}(I_2) = W$. By the Hilbert Basis Theorem there exists $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ such that $I_1 = (f_1, \ldots, f_s)$.
Suppose that $V \subseteq W$. Then for any $f \in I_2$ we have that $f(a_1, \ldots, a_n) = 0$ for all $(a_1, \ldots, a_n) \in W \supseteq V$, so $f \in I_1$. Hence $I_2 \subseteq I_1$. Now suppose $V \subsetneq W$, so that there exists some $(a_1, \ldots, a_n) \in W \setminus V$. Then $f_i(a_1, \ldots, a_n) \neq 0$ for some $1 \leq i \leq s$ (since otherwise $(a_1, \ldots, a_n) \in V$), hence $f_i \notin I_2$, so $I_2 \subsetneq I_1$.
Conversely suppose first that $I_2 \subseteq I_1$. Then for every $(a_1, \ldots, a_n) \in V$ we have that $f(a_1, \ldots, a_n) = 0$ for all $f \in I_1$. Since $I_2 \subseteq I_1$ we see that $(a_1, \ldots, a_n) \in W$ and so $V \subseteq W$. Suppose now that $I_2 \subsetneq I_1$. Then $f_i \notin I_2$ for some $1 \leq i \leq s$ (since otherwise $I_1 = I_2$). But if $V = W$ then $f_i(a_1, \ldots, a_n) = 0$ for all $(a_1, \ldots, a_n) \in V = W$, so $f_i \in I_2$. Hence $V \subsetneq W$.

(b) Let $V_1 \supseteq V_2 \supseteq \ldots$ be a descending chain of affine varieties. Then $\mathbb{I}(V_1) \subseteq \mathbb{I}(V_2) \subseteq \ldots$ is a ascending chain of ideals. But we saw in the proof of the Hilbert Basis Theorem that any such chain stabilises, so that $\mathbb{I}(V_N) = \mathbb{I}(V_{N+1}) = \ldots$ for some $N \geq 1$. By out previous result, so $V_N = V_{N+1} = \ldots$.

(c) Let $I_i := (f_1, \ldots, f_i)$. Then we have an ascending chain of ideals $I_1 \subseteq I_2 \subseteq \ldots$. As observed above, this must eventually stabilise, giving $(f_1, f_2, \ldots) = (f_1, \ldots, f_N)$.

(d) Let $V_i := \mathbb{V}(f_1, \ldots, f_i) \subset k^n$. Then $V_1 \supseteq V_2 \supseteq \ldots$ is a descending chain of affine varieties. By above this stabilises, giving $\mathbb{V}(f_1, f_2, \ldots) = \mathbb{V}(f_1, \ldots, f_N)$.