M3P23, M4P23, M5P23: COMPUTATIONAL ALGEBRA & GEOMETRY
SOLUTIONS 2

(1) (a) Let $f \in \mathbb{C}[x]$, $f \neq 0$. The f factorises completely as

$$f = \beta \prod_{i=1}^{\deg f} (x - \alpha_i)$$

for some $\beta, \alpha_i \in \mathbb{C}$, $\beta \neq 0$. In particular, if $\deg f > 0$ then $f(\alpha_1) = 0$ and so $\alpha_1 \in \mathbb{V}(f) \neq \emptyset$. Conversely if $\deg f = 0$ then $f = \beta$, and since $\beta \neq 0$ we have that $\mathbb{V}(f) = \mathbb{V}(\beta) = \emptyset$.

(b) Let $f_1, \ldots, f_s \in \mathbb{C}[x]$, $f_i \neq 0$. We have that $(f_1, \ldots, f_s) = (f)$, where $f = \gcd\{f_1, \ldots, f_s\}$, hence $\mathbb{V}(f_1, \ldots, f_s) = \emptyset$ if and only if $\mathbb{V}(f) = \emptyset$. By the previous result we see that this happens if and only if f is a constant. Since the gcd is unique up to non-zero scalar multiplication, we have our result.

(c) $\mathbb{R}$ is not an algebraically closed field, so we don't expect (1a) to hold. For example, $x^2 + 1 \in \mathbb{R}[x]$ is a non-zero, non-constant polynomial, but $\mathbb{V}(x^2 + 1) = \emptyset$.

(2) Let

$$f = c \prod_{i=1}^{d} (x - a_i)^{r_i} \in \mathbb{C}[x],$$

where $c, a_i \in \mathbb{C}$, $c \neq 0$, and the a_i distinct. Then $f = 0$ if and only if at least one of the factors $(x - a_i)^{r_i}$ vanishes. But $(x - a_i)^{r_i} = 0$ if and only if $x = a_i$, hence $\mathbb{V}(f) = \{a_1, \ldots, a_d\}$.

Let

$$f_{red} = c \prod_{i=1}^{d} (x - a_i).$$

Then $\mathbb{V}(f_{red}) = \mathbb{V}(f)$ (since the zero-sets coincide). Hence $\mathbb{I}(\mathbb{V}(f)) = \mathbb{I}(\mathbb{V}(f_{red})) \supseteq (f_{red})$. Conversely suppose that $g \in \mathbb{I}(\mathbb{V}(f))$. In particular, $g(a_i) = 0$ for all $1 \leq a_i \leq d$. hence we can factor g as

$$g = h \prod_{i=1}^{d} (x - a_i)$$

for some $h \in \mathbb{C}[x]$. In particular $g = \frac{1}{c} h f_{red}$, so $g \in (f_{red})$. Hence $\mathbb{I}(\mathbb{V}(f)) = (f_{red})$.

(3) (a) Let $f = (x - a)^r h \in \mathbb{C}[x]$, where $h(a) \neq 0$. Differentiating, we get

$$\begin{aligned} f' &= r(x-a)^{r-1} h + (x-a)^r h' \\ &= (x-a)^{r-1} \left(rh + (x-a)h' \right). \end{aligned}$$

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Set $h_1 = rh + (x-a)h' \in \mathbb{C}[x]$. Notice that $h_1(a) = rh(a) + 0 \neq 0$.

(b) Suppose that

$$f = c\prod_{i=1}^{d}(x-a_i)^{r_i}.$$

We proceed by induction on d. When $d = 1$ the result follows from (3a), where $h = c$ is a constant. Suppose that $d > 1$. We can write

$$f = (x-a_d)^{r_d}\left(c\prod_{i=1}^{d-1}(x-a_i)^{r_i}\right).$$

By the proof of (3a), setting

$$h = c\prod_{i=1}^{d-1}(x-a_i)^{r_i}$$

we have

$$f' = (x-a_d)^{r_d-1}\left(r_d h + (x-a_d)h'\right).$$

By the inductive hypothesis we have that

$$h' = H_{d-1}\prod_{i=1}^{d-1}(x-a_i)^{r_i-1}$$

for some $H_{d-1} \in \mathbb{C}[x]$ not vanishing at any a_i, $1 \leq i \leq d-1$. Hence

$$\begin{aligned} f' &= (x-a_d)^{r_d-1}\left(r_d h + (x-a_d)H_{d-1}\prod_{i=1}^{d-1}(x-a_i)^{r_i-1}\right) \\ &= \left(r_d\prod_{i=1}^{d-1}(x-a_i) + (x-a_d)H_{d-1}\right)\prod_{i=1}^{d}(x-a_i)^{r_i-1}. \end{aligned}$$

Set

$$H_d = r_d\prod_{i=1}^{d-1}(x-a_i) + (x-a_d)H_{d-1} \in \mathbb{C}[x]$$

and notice that when $x = a_i$, $1 \leq i \leq d-1$, we have that

$$H_d(a_i) = 0 + (x-a_d)H_{d-1}(a_i) \neq 0$$

(by assumption on H_{d-1}), and when $x = a_d$ we have

$$H_d(a_d) = r_d\prod_{i=1}^{d-1}(a_d-a_i) + 0 \neq 0$$

(since the a_i are distinct).

(c)

$$\begin{aligned}\gcd\{f, f'\} &= \gcd\left\{c\prod_{i=1}^{d}(x-a_i)^{r_i}, H\prod_{i=1}^{d}(x-a_i)^{r_i-1}\right\}\\ &= \left(\prod_{i=1}^{d}(x-a_i)^{r_i-1}\right)\gcd\left\{c\prod_{i=1}^{d}(x-a_i), H\right\}.\end{aligned}$$

Now $\mathbb{V}\left(c\prod_{i=1}^{d}(x-a_i)\right) = \{a_1, \ldots, a_d\}$, and by (3b) we know that H doesn't vanish at any of the a_i. Hence

$$\mathbb{V}\left(c\prod_{i=1}^{d}(x-a_i), H\right) = \mathbb{V}\left(c\prod_{i=1}^{d}(x-a_i)\right) \cap \mathbb{V}(H) = \emptyset.$$

By (1b) this implies that

$$\gcd\left\{c\prod_{i=1}^{d}(x-a_i), H\right\} = 1,$$

so $\gcd\{f, f'\} = \prod_{i=1}^{d}(x-a_i)^{r_i-1}$ as required. In particular,

$$\begin{aligned}\frac{f}{\gcd\{f, f'\}} &= c\prod_{i=1}^{d}\frac{(x-a_i)^{r_i}}{(x-a_i)^{r_i-1}}\\ &= c\prod_{i=1}^{d}(x-a_i)\\ &= f_{red}.\end{aligned}$$

(d) We will use the fact that

$$\mathbb{I}(\mathbb{V}(f)) = (f_{red}) = \left(\frac{f}{\gcd\{f, f'\}}\right) \subset \mathbb{C}[x].$$

(i) $f = x^5 + x^4 - 2x^3 - 2x^2 + x + 1$ and $f' = 5x^4 + 4x^3 - 6x^2 - 4x + 1$.
By Euclid's algorithm we see that

$$\gcd\{f, f'\} = x^3 + x^2 - x - 1.$$

By long division we have that

$$f = (x^2-1)(x^3 + x^2 - x - 1).$$

Hence $\mathbb{I}(\mathbb{V}(f)) = (x^2 - 1)$.

(ii) $f = x^6 + 14x^5 + 80x^4 + 238x^3 + 387x^2 + 324x + 108$ and
$f' = 6x^5 + 70x^4 + 320x^3 + 714x^2 + 774x + 324$.
By Euclid's algorithm we see that

$$\gcd\{f, f'\} = x^3 + 8x^2 + 21x + 18.$$

By long division we have that

$$f = (x^3 + 6x^2 + 11x + 6)(x^3 + 8x^2 + 21x + 18).$$

Hence $\mathbb{I}(\mathbb{V}(f)) = (x^3 + 6x^2 + 11x + 6)$.