M3P23, M4P23, M5P23: COMPUTATIONAL ALGEBRA & GEOMETRY
SHEET 5

(1) Let $f \in k[x_1, \ldots, x_n]$. Recall that a point $a = (a_1, \ldots, a_n) \in k^n$ such that $f(a) = 0$ is said to be *singular* if the partial derivatives

$$\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$$

simultaneously vanish at a.

(a) Show that $(0,0)$ is the only singular point of $y^2 = x^3$.
(b) Find all singular points of the curve $y^2 = cx^2 - x^3$, where c is a constant.
(c) Show that the circle $x^2 + y^2 = a^2$ has no singular points.

(2) Recall that the variety $\mathbb{V}(y - x^2, z - x^3) \subset \mathbb{R}^3$ is called the *twisted cubic*.

(a) Prove that $\mathbb{V}((y - x^2)^2 + (z - x^3)^2)$ is also the twisted cubic.
(b) Let $I \subset \mathbb{R}[x_1, \ldots, x_n]$ be an ideal. Show that any variety $\mathbb{V}(I) \subset \mathbb{R}^n$ can be defined by a single equation.

(3) Given a field k (not necessarily algebraically closed), show that $\sqrt{(x^2, y^2)} = (x, y)$. More generally, show that $\sqrt{(x^n, y^m)} = (x, y)$ for any positive integers n and m.

(4) For the following polynomials f and ideals I, determine whether $f \in \sqrt{I}$. If so, also determine the smallest positive power m such that $f^m \in I$.

(a) $f = x + y$, $I = \big(x^3, y^3, xy(x + y)\big)$.
(b) $f = x^2 + 3xz$, $I = (x + z, x^2y, x - z^2)$.

(5) Prove that if I is a *proper* ideal of $\mathbb{C}[x_1, \ldots, x_n]$ (that is, $I \neq \mathbb{C}[x_1, \ldots, x_n]$), then

$$\sqrt{I} = \bigcap_{\substack{J \text{ prime} \\ I \subseteq J}} J.$$

(6) Let $I = (xz - y^2, z^3 - x^5) \subset \mathbb{C}[x, y, z]$. Express $\mathbb{V}(I) \subset \mathbb{C}^3$ as a finite union of irreducible varieties.

(7) Let $V, W \subset k^n$ be affine varieties such that $V \subset W$. Show that each irreducible component of V is contained in an irreducible component of W.

a.m.kasprzyk@imperial.ac.uk
http://magma.maths.usyd.edu.au/∼kasprzyk/.