M3P23, M4P23, M5P23: COMPUTATIONAL ALGEBRA & GEOMETRY
SHEET 4

(1) (a) Compute a Gröbner basis for

$$I = (x^5 + y^4 + z^3 - 1, x^3 + y^2 + z^2 - 1)$$

using both lex and grevlex orders. Which basis seems preferable?

(b) Repeat this process using the slightly different ideal

$$I = (x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1),$$

where the power of y in the second generator is now 3 rather than 2.

(2) Consider the ideal

$$I_n = (x^{n+1} - yz^{n-1}w, xy^{n-1} - z^n, x^n z - y^n w),$$

where $n \in \mathbb{Z}_{>0}$. Using grevlex order, show that the (reduced) Gröbner basis contains the polynomial

$$z^{n^2+1} - y^{n^2} w$$

when $n = 3, 4$, and 5. (In fact this is true for all n, although you are not asked to prove this.) How big are the Gröbner bases? Repeat the calculate when $n = 3$ using lex order. Any observations?

(3) Consider the system of equations

$$x^2 + 2y^2 = 3, \qquad x^2 + xy + y^2 = 3$$

with corresponding ideal $I \subset k[x, y]$.

(a) Find bases of $I \cap k[x]$ and $I \cap k[y]$.
(b) Find all solutions of the equations.
(c) Which of the solutions are rational (i.e. contained in $\mathbb{Q}^2$)?
(d) What is the smallest field k such that all solutions lie in k^2?

(4) Determine all solutions $(x, y) \in \mathbb{Q}^2$ to the system of equations

$$x^2 + 2y^2 = 2, \qquad x^2 + xy + y^2 = 2.$$

Also determine all solutions in $\mathbb{C}^2$.

(5) Consider the ideal

$$I = (x^2 + y^2 + z^2 - 4, x^2 + 2y^2 - 5, xz - 1).$$

Find bases for the elimination ideals I_1 and I_2. How many rational solutions are there?

`a.m.kasprzyk@imperial.ac.uk`
`http://magma.maths.usyd.edu.au/~kasprzyk/`.

(6) Consider the system of equations

$$x^5 + \frac{1}{x^5} = y, \qquad x + \frac{1}{x} = z.$$

Let $I \subset \mathbb{C}[x, y, z]$ be the ideal defined by these equations.

(a) Find a basis $I_1 \subset \mathbb{C}[y, z]$ and show that $I_2 = \{0\}$.

(b) Use the Extension Theorem to prove that each partial solution in $\mathbb{V}(I_2) = \mathbb{C}$ extends to a solution in $\mathbb{V}(I) \subset \mathbb{C}^3$.

(c) Working over $\mathbb{R}$ now, which partial solutions $(y, z) \in \mathbb{V}(I_1) \subset \mathbb{R}^2$ extend to solutions $\mathbb{V}(I) \subset \mathbb{R}^3$. Why does this not contradict the Extension Theorem?

(d) By regarding z as a parameter, solve for x and y to give a parameterisation of $\mathbb{V}(I) \subset \mathbb{C}^3$.