M3P23, M4P23, M5P23: COMPUTATIONAL ALGEBRA & GEOMETRY

SHEET 3

(1) Let $I = (x^\alpha \mid \alpha \in A) \subset k[x_1, \ldots, x_n]$ be a monomial ideal, where $A \subset \mathbb{Z}^n_{\geq 0}$, and fix a monomial order. Let S be the set of all exponents $\beta \in \mathbb{Z}^n_{\geq 0}$ that occur as a monomial $x^\beta \in I$. Prove that the smallest element in S lies in A.

(2) A set of monomial generators $\{x^{\alpha_1}, \ldots, x^{\alpha_s}\}$ for a monomial ideal I is said to be *minimal* if $x^{\alpha_i} \nmid x^{\alpha_j}$, $i \neq j$. Prove that every monomial ideal has a unique minimal set of generators.

(3) Let $I = (x^{\alpha_1}, \ldots, x^{\alpha_s}) \subset k[x_1, \ldots, x_n]$ be a monomial ideal. For any polynomial $f \in k[x_1, \ldots, x_n]$, prove that $f \in I$ if and only if $\overline{f}^{x^{\alpha_1}, \ldots, x^{\alpha_s}} = 0$.

(4) Compute $S(f, g)$ using lex order, where:
   (a) $f = 4x^2z - 7y^2$, $g = xyz^2 + 3xz^4$.
   (b) $f = x^4y - z^2$, $g = 3xyz^2 - y$.
   (c) $f = xy + z^3$, $g = z^2 - 3z$.

(5) Let $I = (x^4y^2 - z^5, x^3y^3 - 1, x^2y^4 - 2z) \subset \mathbb{C}[x, y, z]$. Using grlex order, is $\{x^4y^2 - z^5, x^3y^3 - 1, x^2y^4 - 2z\}$ a Gröbner basis for I?

(6) Let $f, g \in k[x_1, \ldots, x_n]$ and x^α, x^β be monomials. Show that

$$S(x^\alpha f, x^\beta g) = x^\gamma S(f, g)$$

where

$$x^\gamma = \frac{\operatorname{lcm}\{x^\alpha \operatorname{LM}(f), x^\beta \operatorname{LM}(g)\}}{\operatorname{lcm}\{\operatorname{LM}(f), \operatorname{LM}(g)\}}.$$

(7) (a) Let $V, W \subset k^n$ be affine varieties. Prove that $V \subsetneq W$ if and only if $\mathbb{I}(V) \supsetneq \mathbb{I}(W)$.
   (b) Let

$$V_1 \supset V_2 \supset V_3 \supset \ldots$$

   be a descending chain of affine varieties. Show that there exists some $N \geq 1$ such that $V_N = V_{N+1} = \ldots$
   (c) Let $f_1, f_2, \ldots \in k[x_1, \ldots, x_n]$ be an infinite collection of polynomials and let $I = (f_1, f_2, \ldots)$ be the ideal they generate. Prove that there exists some $N \geq 1$ such that $I = (f_1, \ldots, f_N)$.
   (d) Given polynomials $f_1, f_2, \ldots \in k[x_1, \ldots, x_n]$, let $V = \mathbb{V}(f_1, f_2, \ldots) \subset k^n$. Show that there exists some $N \geq 1$ such that $V = \mathbb{V}(f_1, \ldots, f_N)$.

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