M3P23, M4P23, M5P23: COMPUTATIONAL ALGEBRA & GEOMETRY
SHEET 2

(1) (a) Let $f \in \mathbb{C}[x]$ be a non-zero polynomial. Prove that $\mathbb{V}(f) = \emptyset$ if and only if f is a constant.
(b) Let $f_1, \ldots, f_s \in \mathbb{C}[x]$. Prove that $\mathbb{V}(f_1, \ldots, f_s) = \emptyset$ if and only if $\gcd\{f_1, \ldots, f_s\} = 1$.
(c) What if we replace $\mathbb{C}[x]$ with $\mathbb{R}[x]$?

(2) Let $f \in \mathbb{C}[x]$ be a non-zero polynomial. We can write f as a product of linear factors

$$f = c \prod_{i=1}^{d} (x - a_i)^{r_i}, \tag{1}$$

where $c, a_i \in \mathbb{C}$, $c \neq 0$. Define the *reduced* polynomial of f to be

$$f_{red} = c \prod_{i=1}^{d} (x - a_i), \tag{2}$$

so that $f_{red} \in \mathbb{C}[x]$ is of degree d. Prove that $\mathbb{V}(f) = \{a_1, \ldots, a_d\}$, and that $\mathbb{I}(\mathbb{V}(f)) = (f_{red})$, where $(f_{red}) \subset \mathbb{C}[x]$ is the principal ideal generated by f_{red}.

(3) Throughout let $f \in \mathbb{C}[x]$ be a non-zero polynomial, and let $f' \in \mathbb{C}[x]$ denote the derivative of f.
(a) Suppose that $f = (x - a)^r h \in \mathbb{C}[x]$, where $h(a) \neq 0$. Prove that $f' = (x - a)^{r-1} h_1$ for some $h_1 \in \mathbb{C}[x]$, $h_1(a) \neq 0$.
(b) Now suppose that f is factored completely as in equation (1). Prove that

$$f' = H \prod_{i=1}^{d} (x - a_i)^{r_i - 1},$$

where $H \in \mathbb{C}[x]$ is a polynomial not vanishing at any of the a_i.
(c) Prove that

$$\gcd\{f, f'\} = \prod_{i=1}^{d} (x - a_i)^{r_i - 1}.$$

Hence or otherwise, prove that

$$f_{red} = \frac{f}{\gcd\{f, f'\}},$$

where f_{red} is defined in equation (2).
(d) Calculate $\mathbb{I}(\mathbb{V}(f))$ when:
(i) $f = 1 + x - 2x^2 - 2x^3 + x^4 + x^5 \in \mathbb{C}[x]$;
(ii) $f = 108 + 324x + 387x^2 + 238x^3 + 80x^4 + 14x^5 + x^6 \in \mathbb{C}[x]$.

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