M3P23, M4P23, M5P23: COMPUTATIONAL ALGEBRA & GEOMETRY
SHEET 1

(1) Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a polynomial such that $f(a_1, \ldots, a_n) = 0$ for all $(a_1, \ldots, a_n) \in \mathbb{R}^n$. Prove, by induction on n, that $f = 0$. [Hint: When $n = 1$ we have shown in lectures that f can have at most $\deg f$ roots.]

(2) Let $\mathbb{F}_2 = \mathbb{Z}/(2) = \{0, 1\}$ and define addition and multiplication in the usual way, so that $\mathbb{F}_2$ is a field. Consider $g(x, y) = x^2y + y^2x \in \mathbb{F}_2[x, y]$. Show that $g(a, b) = 0$ for every $(a, b) \in \mathbb{F}_2^2$. Write down a non-zero polynomial in $\mathbb{F}_2[x, y, z]$, involving all three variables x, y, and z, vanishing at every point (a, b, c) in $\mathbb{F}_2^3$. Generalise this to $g \in \mathbb{F}_2[x_1, \ldots, x_n]$ such that $g(a_1, \ldots, a_n) = 0$ for all $(a_1, \ldots, a_n) \in \mathbb{F}_2^n$.

(3) Let $I \subset k[x_1, \ldots, x_n]$ be an ideal, and $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$ be a finite collection of polynomials. Prove that $f_1, \ldots, f_m \in I$ if and only if $(f_1, \ldots, f_m) \subset I$.

(4) Show that $\mathbb{I}(\mathbb{V}(x^n, y^m)) = (x, y)$ for any positive integers n, m.

(5) Let $I \subset \mathbb{F}_2[x, y]$ be the ideal of all polynomials vanishing at all points in $\mathbb{F}_2^2$. Notice that $I \neq \{0\}$ since, for example, the equation $x^2y + y^2x$ in question (2) is an element of I.
   (a) Show that $(x^2 - x, y^2 - y) \subset I$.
   (b) Show that every $f \in \mathbb{F}_2[x, y]$ can be written as
   $$f = A(x^2 - x) + B(y^2 - y) + axy + bx + cy + d,$$
   where $A, B \in \mathbb{F}_2[x, y]$ and $a, b, c, d \in \mathbb{F}_2$. [Hint: Write f in the form $\sum_i p_i(x)y^i$ and use the division algorithm to divide each p_i by $x^2 - x$. From this you can write $f = A(x^2 - x) + q_1(y)x + q_2(y)$. Now divide q_1 and q_2 by $y^2 - y$.]
   (c) Show that $axy + bx + cy + d \in I$ if and only if $a = b = c = d = 0$.
   (d) Conclude that $I = (x^2 - x, y^2 - y)$.
   (e) Express $x^2y + y^2x$ as a combination of $x^2 - x$ and $y^2 - y$. [Hint: Remember that $2 = 1 + 1 = 0$ in $\mathbb{F}_2$.]

(6) Prove that the ideal $(x, y) \subset k[x, y]$ is not principal.

(7) Let $f_1, f_2, \ldots, f_m \in k[x]$ be a finite collection of polynomials, and set $h = \gcd\{f_2, \ldots, f_m\}$. Prove that $(f_1, h) = (f_1, f_2, \ldots, f_m)$.

(8) Use a computer algebra system to compute the following GCDs:
   (a) $\gcd\{x^4 + x^2 + 1, x^4 - x^2 - 2x - 1, x^3 - 1\}$.
   (b) $\gcd\{x^3 + 2x^2 - x - 2, x^3 - 2x^2 - x + 2, x^3 - x^2 - 4x + 4\}$.

(9) Is it true that $x^2 - 4 \in (x^3 + x^2 - 4x - 4, x^3 - x^2 - 4x + 4, x^3 - 2x^2 - x + 2)$?

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