M3P23, M4P23, M5P23: COMPUTATIONAL ALGEBRA & GEOMETRY

REVISION SOLUTIONS

(1) (a) Fix a monomial order. A finite subset $G = \{g_1, \ldots, g_m\}$ of an ideal $I \subset k[x_1, \ldots, x_n]$ is called a *Gröbner basis* if

$$(\mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_m)) = (\mathrm{LT}(I)),$$

where $\mathrm{LT}(g_i)$ is the leading term of the polynomial g_i with respect to the monomial order, and

$$(\mathrm{LT}(I)) = (\mathrm{LT}(f) \mid f \in I)$$

is the monomial ideal generated by the leading terms of all polynomials f in I.
Given two polynomials $f, g \in k[x_1, \ldots, x_n]$, the *S-polynomial* $S(f, g)$ is defined by

$$S(f,g) = \frac{x^\alpha}{\mathrm{LT}(f)} f - \frac{x^\alpha}{\mathrm{LT}(g)} g,$$

where $x^\alpha = \mathrm{lcm}\{\mathrm{LM}(f), \mathrm{LM}(g)\}$ is the monomial in $k[x_1, \ldots, x_n]$ given by the least common multiple of the leading monomials of f and g.
Given a finite set of generators $G = \{g_1, \ldots, g_m\}$ for an ideal I, G can be transformed into a Gröbner basis as follows: We calculate the remainder $\overline{S(g_i, g_j)}^G$ of $S(g_i, g_j)$ upon division by G, for each $i \neq j$; when the remainder is non-zero, we include it in the set G to obtain a new set of generators G' for I and repeat. After a finite number of steps, this process will stabilise with a set G'' all of whose S-polynomials have remainder zero upon division by G''. By Buchberger's Criterion, G'' is a Gröbner basis for I.

(b) Let $G = \{x^2 - y, x^4 - 2x^2y\}$. We have that

$$S(x^2 - y, x^4 - 2x^2y) = x^4 - x^2y - x^4 + 2x^2y = x^2y,$$

and $\overline{x^2y}^G = y^2$. So we set $G' = \{x^2 - y, x^4 - 2x^2y, y^2\}$.
We know by construction that $\overline{S(x^2 - y, x^4 - 2x^2y)}^{G'} = 0$. We consider the remaining two S-polynomials.

$$S(x^2 - y, y^2) = x^2y^2 - y^3 - y^2x^2 = -y^3, \quad \text{and } \overline{-y^3}^{G'} = 0$$

$$S(x^4 - 2x^2y, y^2) = x^4y^2 - 2x^2y^3 - x^4y^2 = -2x^2y^3, \quad \text{and } \overline{-2x^2y^3}^{G'} = 0.$$

Hence, by Buchberger's Criterion, G' is a Grönber basis.

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(c) A Gröbner basis G is said to be *minimal* if $\mathrm{LC}(g) = 1$ and $\mathrm{LT}(g) \notin (\mathrm{LT}(G \setminus \{g\}))$, for all $g \in G$. Our Gröbner basis in (b) is not minimal, since

$$\mathrm{LT}\big(x^4 - 2x^2y\big) = x^4 \in \big(\mathrm{LT}\big(x^2 - y\big), \mathrm{LT}\big(y^2\big)\big) = (x^2, y^2).$$

A Gröbner basis G is said to be *reduced* if $\mathrm{LC}(g) = 1$ and no monomial of g is contained in $(\mathrm{LT}(G \setminus \{g\}))$, for all $g \in G$. Clearly any reduced Gröbner basis is minimal, so our Gröbner basis in (b) is not reduced.

We can transform $G = \{x^2 - y, x^4 - 2x^2y, y^2\}$ into a reduced Gröbner basis in a finite number of steps as follows: For each $g \in G$ we calculate $g' = \overline{g}^{G \setminus \{g\}}$ and set $G' = (G \setminus \{g\}) \cup \{g'\}$. The resulting set G' is still a Gröbner basis for I. Repeating until this process stabilises, we obtain (by construction) the reduced Gröbner basis for I.

$$\overline{x^2 - y}^{x^4 - 2x^2y, y^2} = x^2 - y$$
$$\overline{x^4 - 2x^2y}^{x^2 - y, y^2} = 0$$

so we have $G' = \{x^2 - y, y^2\}$, and

$$\overline{x^2 - y}^{y^2} = x^2 - y$$
$$\overline{y^2}^{x^2 - y} = y^2.$$

Hence $G' = \{x^2 - y, y^2\}$ is the reduced Gröbner basis for I.

(2) (a) We define the i-th *elimination ideal* of $I \subset k[x_1, \ldots, x_n]$ to be the ideal in $k[x_{i+1}, \ldots, x_n]$ given by

$$I_i := I \cap k[x_{i+1}, \ldots, x_n].$$

Let G be a Gröbner basis for I with respect to the usual lex order. Then, by the Elimination Theorem,

$$G_i := G \cap k[x_{i+1}, \ldots, x_n]$$

is a Gröbner basis for I_i.

A lex-ordered Gröbner basis for the ideal $I \subset \mathbb{C}[x, y, z]$ generated by the given system of equations is

$$G = \{x^2 - y - z, y^3 + yz - 2y - z + 1, z^2 - 3z + 2\}.$$

Hence $G_2 = \{z^2 - 3z + 2\}$ is a Gröbner basis for $I_2 = I \cap \mathbb{C}[z]$, so $\mathbb{V}(I_2) = \{1, 2\} \subset \mathbb{C}$. We use these two partial solutions to calculate

$$\mathbb{V}(I_1) = \mathbb{V}(y^3 + yz - 2y - z + 1, z^2 - 3z + 2) \subset \mathbb{C}^2.$$

When $z = 1$ we have $y(y^2 - 1) = 0$, with solutions $y = 0, \pm 1$. When $z = 2$ we have $y^3 = 1$, with solutions $y = 1, \zeta, \zeta^2$. Hence

$$\mathbb{V}(I_1) = \big\{(0, 1), (-1, 1), (1, 1), (1, 2), (\zeta, 2), (\zeta^2, 2)\big\} \subset \mathbb{C}^2.$$

Finally, we lift these partial solutions to find all solutions of $\mathbb{V}(I) \subset \mathbb{C}^3$. We have, for each partial solution respectively,

$$\begin{aligned} x^2 &= 1 \Rightarrow x = \pm 1, \\ x^2 &= 0 \Rightarrow x = 0, \\ x^2 &= 2 \Rightarrow x = \pm\sqrt{2}, \\ x^2 &= 3 \Rightarrow x = \pm\sqrt{3}, \\ x^2 &= 2+\zeta \Rightarrow x = \pm\sqrt{2+\zeta}, \\ x^2 &= 2+\zeta^2 \Rightarrow x = \pm\sqrt{2+\zeta^2}. \end{aligned}$$

Hence the system of equations has eleven solutions, given by the points

$$(\pm 1, 0, 1),\ (0, -1, 1),\ (\pm\sqrt{2}, 1, 1),$$
$$(\pm\sqrt{3}, 1, 2),\ (\pm\sqrt{2+\zeta}, \zeta, 2),\ (\pm\sqrt{2+\zeta^2}, \zeta^2, 2).$$

(b) We begin by computing a Gröbner basis for $I = (x^2z - y^2, yx - x + 1)$ with respect to the usual lex order:

$$G = \{xy - x + 1, xz + y^3 - y^2, y^4 - 2y^3 + y^2 - z\}.$$

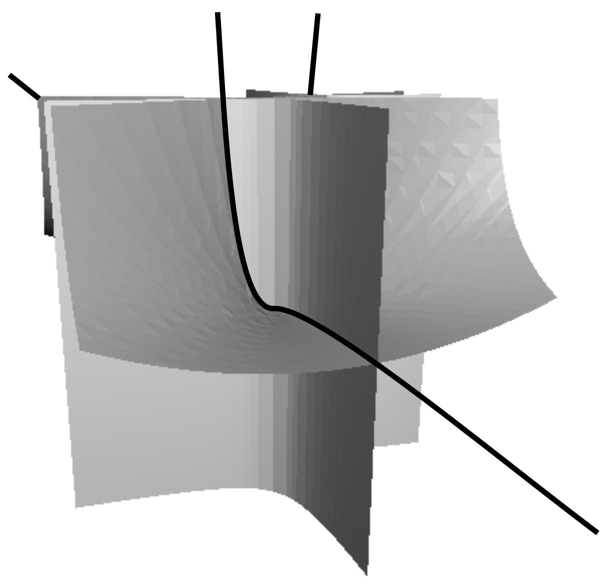

FIGURE 1. The curve cut out by the intersection of the surfaces $x^2z - y^2 = 0$ and $yx - x + 1 = 0$ in $\mathbb{R}^3$.

As in (a) we make use of the Elimination Theorem. Notice that I_2 is the zero ideal, so be begin by considering $I_1 \subset \mathbb{R}[y, z]$. This has Gröbner basis $G_2 = \{y^4 - 2y^3 + y^2 - z\}$, so

$$\mathbb{V}(I_1) = \{(t, t^4 - 2t^3 + t^2) \mid t \in \mathbb{R}\}.$$

We now attempt to lift these partial solutions. First we consider the equation $xz + y^3 - y^2 = 0$. We obtain:

$$\begin{aligned}
x(t^4 - 2t^3 + t^2) + t^3 - t^2 &= 0 \\
\Rightarrow t^2\left(x(t-1)^2 + (t-1)\right) &= 0 \\
&\Rightarrow t = 0 \text{ and } x \text{ is free} \\
&\text{or } t = 1 \text{ and } x \text{ is free} \\
&\text{or } t \neq 0, 1 \text{ and } x = \frac{1}{1-t}.
\end{aligned}$$

Now we consider the equation $xy - x + 1 = 0$. This tells us that $t \neq 1$, and that when $t \neq 1$ we have $x = \frac{1}{1-t}$. Combining these results we find that:

$$\mathbb{V}(x^2z - y^2, yx - x + 1) = \left\{ \left(\frac{1}{1-t}, t, t^4 - 2t^3 + t^2 \right) \;\middle|\; t \in \mathbb{R} \setminus \{1\} \right\}.$$

The curve is illustrated in Figure 1.

Let $\pi_1 : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection map along the x-axis onto the (y, z)-plane. The image $\pi_1(C)$ of the curve $C = \mathbb{V}(x^2z - y^2, yx - x + 1) \subset \mathbb{R}^3$ is

$$\left\{(t, t^4 - 2t^3 + t^2) \mid t \in \mathbb{R} \setminus \{1\}\right\} \subset \mathbb{R}^2,$$

where the point $(1, 0)$, corresponding to $t = 1$, has been removed. This is illustrated in Figure 2. The Closure Theorem tells us that

$$\mathbb{V}(I_1) = \{(t, t^4 - 2t^3 + t^2) \mid t \in \mathbb{R}\} \supset \pi_1(C)$$

is the smallest affine variety containing $\pi_1(C)$.

(3) (a) An ideal $I \subset k[x_1, \ldots, x_n]$ is called a *monomial ideal* if there exists a (possibly infinite) subset $\Lambda \subset \mathbb{Z}^n_{\geq 0}$ such that

$$I = \left\{ \sum_{\alpha \in \Lambda} h_\alpha x^\alpha \;\middle|\; h_\alpha \in k[x_1, \ldots, x_n] \text{ and finitely many } h_\alpha \neq 0 \right\}.$$

We write $I = (x^\alpha \mid \alpha \in \Lambda)$.

Suppose that x^β is a multiple of x^α for some $\alpha \in \Lambda$. Then $x^\beta \in I$ by definition of an ideal. Conversely suppose that $x^\beta \in I$. Then

$$x^\beta = \sum_{i=1}^{m} x^{\alpha_i} h_i, \qquad \text{where } \alpha_i \in \Lambda \text{ and } h_i \in k[x_1, \ldots, x_n].$$

Expand each h_i as a linear combination of monomials. We see that every term on the right-hand side of the expression is divisible by some x^{α_i}. Hence the left-hand side must also be divisible by some x^α, $\alpha \in \Lambda$, and we are done.

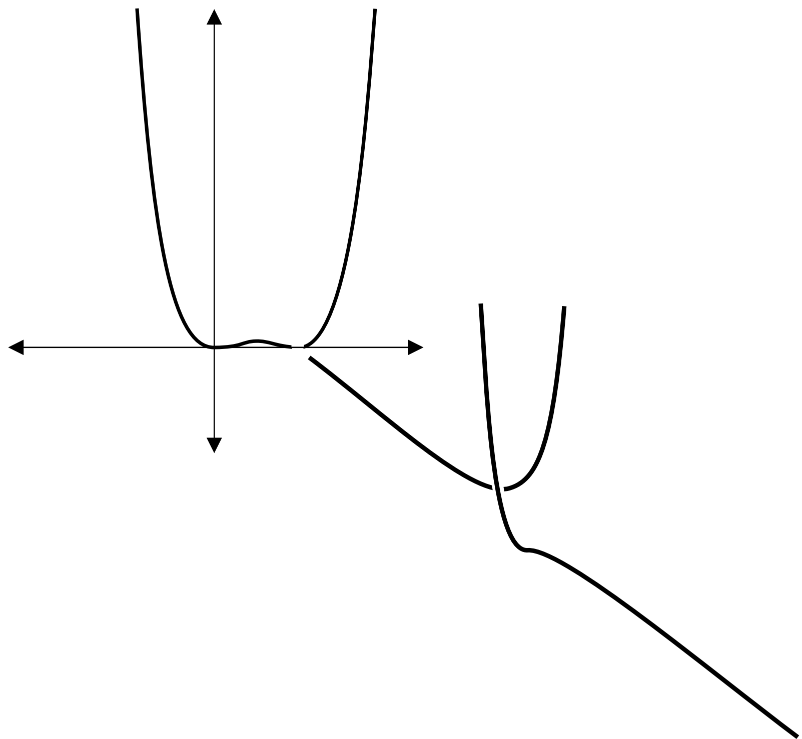

FIGURE 2. The projection of the curve onto the (x, y)-plane. Notice how the image is missing the point $(1, 0)$ corresponding to $t = 1$ in the parameterisation.

(b) The monomials in J are sketched in Figure 3. The terms appearing in the remainder upon performing the division algorithm are given by k-linear sums of monomials outside of the shaded region; i.e. $\sum_{\beta\in\Gamma} c_\beta x^\beta$ where $c_\beta \in k$ and

$$\begin{aligned}\Gamma := &\{(i,j) \mid i \in \{0,1\}, j \in \{0,1,2,3,4\}\} \cup \\ &\{(2,j) \mid j \in \{0,1,2\}\} \cup \\ &\{(i,j) \mid i \in \{3,4\}, j \in \{0,1\}\} \cup \\ &\{(5,0)\}\,.\end{aligned}$$

(c) Using lex order we have remainder $2x^3 + 3x^2y$.

(4) (a) Let $\tilde{I} = (f_1, \ldots, f_r, 1 - wf) \subset k[x_1, \ldots, x_n]$. From the proof of the Nullstellensatz we have that $1 \in \tilde{I}$ only if $f^m \in I$ for some m. Hence $f \in \sqrt{I}$. Conversely suppose that $f \in \sqrt{I}$. Then $f^m \in I$ for some m, and so $f^m \in \tilde{I}$. Since $1 - wf \in \tilde{I}$ by definition, we see that

$$\begin{aligned}1 &= w^m f^m + (1 - w^m f^m) \\ &= w^m f^m + (1 - wf)(1 + wf + w^2 f^2 + \ldots + w^{m-1} f^{m-1}) \in \tilde{I}.\end{aligned}$$

(b) Using lex order, the reduced Gröbner basis for $\tilde{I}$ is $\{1\}$. Hence, by (a), $f \in \mathbb{I}$. To determine the smallest power m such that $f^m \in I$ we begin by calculating the

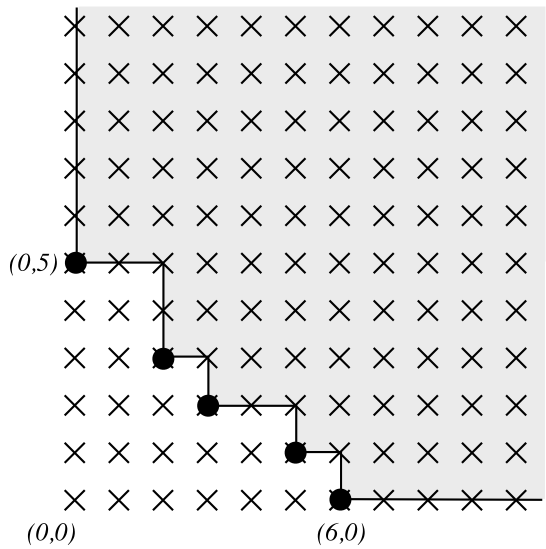

FIGURE 3. The monomials in the ideal J, where $x^a y^b \leftrightarrow (a,b) \in \mathbb{Z}^2_{\geq 0}$.

reduced Gröbner basis for I (again using lex order):

$$G = \{x^4 - 2x^2 + 1, y^2\}.$$

We now calculate the remainder $\overline{f^m}^G$ for successive values of m:

$$\begin{aligned}\overline{f}^G &= -x^2 + y + 1,\\ \overline{f^2}^G &= -2x^2 y + 2y,\\ \overline{f^3}^G &= 0.\end{aligned}$$

Hence $m = 3$.

(c) Define

$$f_{red} := \frac{f}{\gcd\left\{f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\}}.$$

Then $\sqrt{I} = (f_{red})$. We find that $f_{red} = xy(x^2 + y - 1) + y^2$, and so

$$\sqrt{I} = \left(xy(x^2 + y - 1) + y^2\right).$$

(5) (a) We begin by calculating the partial derivatives:

$$\begin{aligned}\frac{\partial f}{\partial x} &= 8x^3(x^4 + y^4) - 2xy^2,\\ \frac{\partial f}{\partial y} &= 8y^3(x^4 + y^4) - 2x^2 y.\end{aligned}$$

The ideal $I = \left((x^4 + y^4)^2 - x^2y^2, 8x^3(x^4 + y^4) - 2xy^2, 8y^3(x^4 + y^4) - 2x^2 y\right)$ has reduced Gröbner basis

$$G = \left\{x^7 - \frac{xy^2}{4}, x^2 y - 4y^7, xy^3, y^8\right\}$$

with respect to the usual lex order. By the Elimination Theorem we have partial solution $\mathbb{V}(y^8) = \{0\}$; i.e. $y = 0$. Extending this we see that $x = 0$. We conclude from the definition of singularity that $f = 0$ has only one singular point, when $x = y = 0$.

(b) In this case the partial derivatives are:

$$\frac{\partial g}{\partial x} = -6xz^4 + 2y^5,$$
$$\frac{\partial g}{\partial y} = 10xy^4 + 10yz^4,$$
$$\frac{\partial g}{\partial z} = -12x^2z^3 + 20y^2z^3.$$

We obtain the lex-ordered reduced Gröbner basis

$$G = \left\{x^2z^3 - \frac{5y^2z^3}{3}, xy^4 + yz^4, xz^4 - \frac{y^5}{3}, y^7, y^6z^3, y^2z^4, yz^8\right\}.$$

We have the partial solutions $\mathbb{V}(y^7, y^6z^3, y^2z^4, yz^8) = \mathbb{V}(y)$; i.e. there is a line of partial solutions parameterised by $(0, t)$. We now attempt to lift these solutions. First we consider the equation $xz^4 - \frac{y^5}{3} = 0$. This gives us $xt^4 = 0$, implying that either $t = 0$ or $x = 0$. The equation $xy^4 + yz^4 = 0$ tells us nothing new (since $y = 0$). Similarly for $x^2z^3 - \frac{5y^2z^3}{3} = 0$. Hence we see that

$$\mathbb{V}(I) = \mathbb{V}(y, z) \cup \mathbb{V}(x, y) = \{(s, 0, 0)\} \cup \{(0, 0, t)\},$$

so there are two lines of singularities, corresponding to the x-axis and to the z-axis.