M3P23, M4P23, M5P23: COMPUTATIONAL ALGEBRA & GEOMETRY

REVISION PROBLEMS

Unless stated otherwise, you may use a computational algebra package to aid your calculations. However, you must be clear in your own mind about how the result has been calculated.

(1) (a) State clearly the definition of *Gröbner basis*, and the definition of *S-polynomial*. Explain how the S-polynomial is used when calculating a Gröbner basis.

(b) Without the aid of a computer, find a Gröbner basis for the ideal

$$(x^2 - y, x^4 - 2x^2y)$$

using lex order. Be sure that you explain how you know that your answer is a Gröbner basis.

(c) Is your Gröbner basis minimal? Is it reduced? If not, adjust it so that it is.

(2) (a) Find all possible solutions to the system of equations

$$\begin{cases} x^{10} - 22x^6 + 51x^4 - 48x^2 + 18 = -18y, \\ x^{10} - 22x^6 + 51x^4 - 30x^2 + 18 = 18z, \\ x^{12} - 9x^{10} + 32x^8 - 57x^6 + 51x^4 - 18x^2 = 0. \end{cases}$$

If appropriate, write your solutions in terms of ζ and ζ^2, where $1, \zeta$ and ζ^2 are the three cube roots of unity.

(b) Find a parameterisation for the curve

$$\mathbb{V}(x^2z - y^2, yx - x + 1) \subset \mathbb{R}^3.$$

What is the image of its projection along the x-axis onto the (y, z)-plane? In the image of this projection, explain what is happening at the point $(1, 0)$.

(3) (a) State the definition of *monomial ideal*. Prove that if

$$I = (x^\alpha \mid \alpha \in A)$$

is a monomial ideal, then a monomial $x^\beta \in I$ if and only if x^β is divisible by x^α for some $\alpha \in A$.

(b) Sketch a picture of the monomials contained in the ideal

$$J = (x^6, x^5y, x^3y^2, x^2y^3, y^5) \subset k[x, y].$$

If we perform the division algorithm on some $f \in k[x, y]$ using the generators of J as divisors, write down explicitly which terms can appear in the remainder.

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(c) Making sure that you state which monomial order you are using, calculate the remainder when $f = 2x^3 + x^7y^2 + 3x^2y + y^6$.

(4) (a) Let $I = (f_1, \ldots, f_r)$ be an ideal in $\mathbb{C}[x_1, \ldots, x_n]$. Prove that $f \in \sqrt{I}$ if and only if $1 \in (f_1, \ldots, f_r, 1 - wf) \subset \mathbb{C}[x_1, \ldots, x_n, w]$.

(b) If $I = (xy^2 - 2y^2, x^4 - 2x^2 + 1) \subset \mathbb{C}[x, y]$ and $f = y - x^2 + 1$, is $f \in \sqrt{I}$? If so, what is the smallest power $m > 0$ of f such that $f^m \in I$?

(c) Let

$$I = \Big((x^7 - x^6 - 2x^5 + 2x^4 + x^3 - x^2)y^2 + (2x^5 - 4x^4 + 4x^2 - 2x)y^3 + (x^3 - 3x^2 + 3x - 1)y^4\Big) \subset \mathbb{C}[x, y]$$

be a principal ideal. Find $\sqrt{I}$.

(5) Let $f \in k[x_1, \ldots, x_n]$ be a polynomial. We say that f has a *singularity* at the point $(a_1, \ldots, a_n) \in k^n$ if its partial derivatives all vanish at that point; i.e. if:

$$\frac{\partial f}{\partial x_1}(a_1, \ldots, a_n) = \ldots = \frac{\partial f}{\partial x_n}(a_1, \ldots, a_n) = 0.$$

(a) Let $f = (x^4 + y^4)^2 - x^2y^2 \in \mathbb{R}[x, y]$. By performing a suitable Gröbner basis calculation, show that

$$\mathbb{V}\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \{(0, 0)\}.$$

Hence or otherwise conclude that $f = 0$ has only one singular point, at the origin.

(b) Show that $g = 2xy^5 + 5y^2z^4 - 3x^2z^4$ has two lines of singularities. What are they?