M3P23, M4P23, M5P23: COMPUTATIONAL ALGEBRA & GEOMETRY
EXAM

(1) (a) State the definition of a *reduced* Gröbner basis. Describe an important application of the reduced Gröbner basis when comparing two polynomial ideals $I_1, I_2 \subset k[x_1, \ldots, x_n]$.
(b) State Buchberger's Criterion and, using grlex order, verify that
$$G = \left\{x^2 + 2xy, xy, y^2 - \frac{x}{2}\right\} \subset k[x, y]$$
is a Gröbner basis.
(c) Find a reduced Gröbner basis for the ideal
$$I = \left(x^2 + 2xy, xy, y^2 - \frac{x}{2}\right) \subset k[x, y],$$
using grlex order.

(2) (a) Fix a monomial ordering and let $I \subset k[x_1, \ldots, x_n]$ be an ideal. Let $f \in k[x_1, \ldots, x_n]$ be a polynomial.
(i) Prove that f can be written in the form $f = g + r$, where $g \in I$ and no term of r is divisible by an element of $\mathrm{LT}(I)$.
[Hint: Use the Division Algorithm.]
(ii) Given two expressions $f = g + r = g' + r'$ as in (i) above, prove that $r = r'$.
(b) Let G and G' be two Gröbner bases for an ideal I with respect to the *same* monomial order. Show that $\overline{f}^G = \overline{f}^{G'}$ for all $f \in k[x_1, \ldots, k_n]$.
(c) *Without calculating any S-polynomials*, prove that $G = \{y - x^2, z - x^3\}$ is not a Gröbner basis with respect to lex order.

(3) (a) State the definition of *radical ideal*. Given an ideal $I \subset k[x_1, \ldots, x_n]$, prove that $\sqrt{I}$ is an ideal.
(b) Calculate $\sqrt{(x^2 + y^2 + 2xy, x^2 + y^2 - 2xy)} \subset k[x, y]$.
(c) Let $I = \big(x^2 - 1, y(x + 2)\big) \subset \mathbb{C}[x, y]$. By computing $\mathbb{V}(I) \subset \mathbb{C}^2$, write down $\sqrt{I}$. Hence or otherwise show that $y - x^2 + 1 \in \sqrt{I}$.

(4) (a) Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. State the definition of a *prime ideal* and *maximal ideal*. Prove that if I is a maximal ideal then I is a prime ideal.
(b) Let $I = (x^2 + 1) \subset k[x]$. Show that I is maximal when $k = \mathbb{R}$, but that I is not maximal when $k = \mathbb{C}$.
(c) Let $I \subset \mathbb{R}[x_1, \ldots, x_n]$ be a maximal ideal. Prove that $\mathbb{V}(I) \subset \mathbb{R}^n$ is equal to either the empty set $\emptyset$ or a point $(a_1, \ldots, a_n) \in \mathbb{R}^n$.

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