Birational involutions of the projective plane

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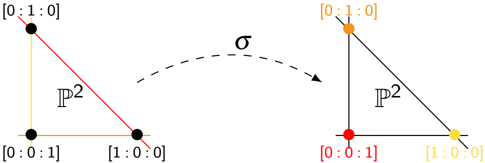
Online Nottingham algebraic geometry seminar

The Cremona group of rank n over a field $\mathbb{K}$ is the group $\mathrm{Cr}_n(\mathbb{K}) = \mathrm{Bir}(\mathbb{P}^n_{\mathbb{K}})$ of birational automorphisms of the projective n-space over $\mathbb{K}$.

In algebraic language, $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{K}}) \simeq \mathrm{Aut}_{\mathbb{K}}\,\mathbb{K}(x_1, \dots, x_n)$.

The most famous example: the Cremona involution of $\mathbb{P}^2$:

$$\begin{array}{cccc} \sigma : & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^2 \\ & [x : y : z] & \mapsto & [yz : xz : xy]. \end{array}$$



Note that $\mathrm{Ind}(\sigma) = \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$.

Examples of birational maps of $\mathbb{P}^2_{\mathbb{C}}$

- $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}}) \simeq \mathrm{PGL}_3(\mathbb{C}) \subset$ is a subgroup of $\mathrm{Cr}_2(\mathbb{C})$.
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  In fact, there is a remarkable

  Theorem (Noether-Castelnuovo, 1871) One has

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  Corollary: $\mathrm{Cr}_2(\mathbb{C})$ is generated by **involutions**.

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Now look at the birational map

$$\sigma_1 : [x : y : z] \mapsto [y^2 + z^2 : xy : xz].$$

Then $\mathrm{Ind}(\psi)$ is not real, as ψ is not defined e.g. in $[0 : 1 : \pm i]$.
Therefore, $\mathrm{Cr}_2(\mathbb{R})$ cannot be generated only by $\mathrm{PGL}_3(\mathbb{R})$ and σ_0.

Examples of birational maps of $\mathbb{P}^2$

$$\sigma_0 : [x:y:z] \mapsto [yz:xz:xy], \quad \sigma_1 : [x:y:z] \mapsto [y^2+z^2:xy:xz].$$

- Let p_1, $\overline{p_1}$, p_2, $\overline{p_2}$, p_3, $\overline{p_3}$ be three pairs of imaginary points of $\mathbb{P}^2_{\mathbb{R}}$, not lying on the same conic. Denote by $f : X \to \mathbb{P}^2_{\mathbb{R}}$ the blow-up of the six points. Note that X is isomorphic to a smooth cubic of $\mathbb{P}^3_{\mathbb{R}}$.

  The set of strict transforms of the conics passing through five of the six points corresponds to three pairs of imaginary lines, and the six curves are disjoint. The contraction of the six curves gives a birational morphism $g : X \to \mathbb{P}^2_{\mathbb{R}}$. The map $g \circ f^{-1}$ is called a standard quintic transformation of $\mathbb{P}^2_{\mathbb{R}}$.

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Theorem (J. Blanc – F. Mangolte, 2012) The group $\mathrm{Cr}_2(\mathbb{R})$ is generated by $\mathrm{PGL}_3(\mathbb{R})$, σ_0, σ_1 and standard quintic involutions.

With a bit of more work, one can deduce from this theorem the following

Corollary (S. Zimmermann): $\mathrm{Cr}_2(\mathbb{R})$ is generated by involutions.

Theorem (S. Lamy – J. Schneider, 2021) The Cremona group $\mathrm{Cr}_2(\mathbb{K})$ is generated by involutions for every perfect field $\mathbb{K}$.

More examples

- Let $p_1, \ldots, p_7 \in \mathbb{P}^2_{\mathbb{C}}$ be seven points in general position, and $\mathscr{L}$ be the linear system of cubics passing through p_i. Then $\dim \mathscr{L} = 2$.

  Let $p \in \mathbb{P}^2_{\mathbb{C}}$ be a general point and $\mathscr{L}_p = \{L \in \mathscr{L} : p \in L\}$ be a pencil of cubics passing through p. Recall that it has 9 base points. Define a map

  $$p \mapsto \gamma(p) := \text{9th base point.}$$

  One can show that this is a birational involution of $\mathbb{P}^2_{\mathbb{C}}$. It is called the Geiser involution.

- Similarly, let $p_1, \ldots, p_8 \in \mathbb{P}^2_{\mathbb{C}}$ be eight points in general position, and $\mathscr{C}$ be the pencil of cubics passing through p_i. Let p_9 be the 9th base point of this pencil.

  Let $p \in \mathbb{P}^2_{\mathbb{C}}$ be a general point. There is a unique cubic $C_p \in \mathscr{C}$ passing through p. Define a map

  $$p \mapsto \beta(p) := -p,$$

  where $-$ is taken with respect to the group law on $\mathscr{C}_p$ with $p_9 = 0$. One can show that this is a birational involution of $\mathbb{P}^2_{\mathbb{C}}$. It is called the Bertini involution.

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Theorem (L. Bayle – A. Beauville, 2000) Every involution in $Cr_2(\mathbb{C})$ is conjugate to one of the following:

- Linear involution of $\mathbb{P}^2_{\mathbb{C}}$.
- Geiser involution.
- Bertini involution.
- de Jonquières involution: locally it is given by

$$(x,y) \mapsto \left(x, \frac{P(x)}{y}\right),$$

  where $P(x)$ is a polynomial of degree $2g+1$ with no multiple roots.

Remark: Geiser, Bertini and de Jonquières involutions have moduli (these are 3 families of involutions in fact). We will see this later.

So, there are two things to explain:

- First, why every involution is conjugate in $\mathrm{Cr}_2(\mathbb{C})$ to one of these four?
- Second, why these four types are actually different (i.e. pairwise non-conjugate)?

Let me start with the first question.

G-surfaces

Let me briefly recall a general strategy of classification of finite subgroups in $\mathrm{Cr}_2(\mathbb{K})$. It is based on the following observations:

- For any finite subgroup $G \subset \mathrm{Cr}_2(\mathbb{K})$ there exists a $\mathbb{K}$-rational smooth projective surface X, an injective homomorphism $\iota : G \to \mathrm{Aut}_{\mathbb{K}}(X)$ and a birational G-equivariant $\mathbb{K}$-map $\psi : X \dashrightarrow \mathbb{P}^2_{\mathbb{K}}$, such that

$$G = \psi \circ \iota(G) \circ \psi^{-1}$$

  This process of passing from a birational action of G on $\mathbb{P}^2_{\mathbb{K}}$ to a regular action on X is usually called the regularization of the G-action. On the other hand, for a $\mathbb{K}$-rational G-surface X a birational map $\psi : X \dashrightarrow \mathbb{P}^2_{\mathbb{K}}$ yields an injective homomorphism

$$i_\psi : G \to \mathrm{Cr}_2(\mathbb{K}), \quad g \mapsto \psi \circ g \circ \psi^{-1}.$$

Moreover, two subgroups of $\mathrm{Cr}_2(\mathbb{K})$ are conjugate if and only if the corresponding G-surfaces are G-birationally equivalent.

So, there is a natural bijection

Conjugacy classes of finite subgroups $G \subset \mathrm{Cr}_2(\mathbb{K})$	$\longleftrightarrow$	birational isomorphism classes of smooth $\mathbb{K}$-rational G-surfaces (X, G).

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Of course, using G-MMP, we can choose a minimal representative in each birational class.

Good news: geometrically rational minimal G-surfaces are completely classified!

Theorem (V. Iskovskikh) Let $\mathbb{K}$ be a perfect field, X be a G-minimal surface which is rational over $\overline{\mathbb{K}}$. Then one of the following holds:

1) X admits a conic bundle structure with $\mathrm{Pic}(X)^G \cong \mathbb{Z}^2$;
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Moral: classification of finite subgroups $G \subset \mathrm{Cr}_2(\mathbb{K})$ up to conjugacy is equivalent to birational classification of the pairs (X, G), where X is as in Iskovskikh's theorem, and moreover X is $\mathbb{K}$-rational.

Over $\mathbb{K} = \mathbb{C}$, the application of this program leads to the proof of the Bayle-Beauville's theorem.

Let $G = \langle \tau \rangle$ be a group of order 2. Then it is not difficult to show that the only G-minimal complex del Pezzo surfaces are:

- a del Pezzo surface S of degree 2 with $\tau \in \mathrm{Aut}(S)$ being the Geiser involution. Recall that S is the double cover

$$\varphi_{|-K_S|} : S \to \mathbb{P}^2_{\mathbb{C}},$$

branched over a smooth quartic $B \subset \mathbb{P}^2_{\mathbb{C}}$. The Geiser involution is the Galois involution of this double cover.

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- a del Pezzo surface S of degree 1 with $\tau \in \mathrm{Aut}(S)$ being the Bertini involution. Recall that S is the double cover

  $$\varphi_{|-2K_S|} : S \to Q,$$

  where $Q \subset \mathbb{P}^3$ is the quadratic cone (branched along a sextic curve). The Bertini involution is exactly the Galois involution of this double cover.

Now let me pass to the second question: how to prove that all these involutions (linear, Geiser, Bertini, de Jonquières) are not conjugate in $\mathrm{Cr}_2(\mathbb{C})$?

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Let $\tau \in \mathrm{Bir}(S)$ and $\tau' \in \mathrm{Bir}(S')$ be two involutions, and assume there is a conjugating map birational map $\varphi : S \dashrightarrow S'$:

$$\begin{array}{ccc} S & \overset{\tau}{\dashrightarrow} & S \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ S' & \overset{\tau'}{\dashrightarrow} & S' \end{array}$$

If τ fixes a curve C then $\tau' = \varphi \circ \tau \circ \varphi^{-1}$ fixes $\varphi(C) \simeq C$.

Therefore,

$$F(\tau) = \bigcup_{C \not\approx \mathbb{P}^1 \text{ is fixed by } \tau} C,$$

is an invariant of the equivalence class of τ!

Now:

- If τ is linear, then $F(\tau) \simeq \mathbb{P}^1$.
- If τ is the Geiser involution, then $F(\tau)$ is a non-hyperelliptic genus 3 curve (plane quartic).
- If τ is the Bertini involution, then $F(\tau)$ is a genus 4 curve of degree 6.
- If τ is the de Jonquières involution

$$(x, y) \mapsto \left(x, \frac{P(x)}{y}\right),$$

then $F(\tau)$ is the hyperelliptic curve $\{y^2 = P(x)\}$.

We conclude, that our 4 types of involutions are indeed pairwise non-conjugate.

In fact, this correspondence:

conjugacy class of an involution $\mapsto$ isom. class of the fixed curve

is bijective. So, there is a one-to-one correspondence between

- conjugacy classes of Geiser involutions and isomorphism classes of non-hyperelliptic curves of genus 3;
- conjugacy classes of Bertini involutions and isomorphism classes of non-hyperelliptic curves of genus 4 whose canonical model lies on a singular quadric.
- de Jonquières involutions of degree d and hyperelliptic curves of genus $d-2$.

This was the story over $\mathbb{C}$. What if we want to classify involutions in $\mathrm{Cr}_2(\mathbb{K})$ for other fields $\mathbb{K}$?

In the remaining part of the talk, I will report on the joint work (in progress) with I. Cheltsov, F. Mangolte and S. Zimmermann.

We classify involutions in $\mathrm{Cr}_2(\mathbb{R})$.

$$\mathrm{Cr}_2(\mathbb{R})$$

Some remarks

- All finite subgroups of $Cr_2(\mathbb{C})$ were classified in 2006 by J. Blanc (abelian case), I. Dolgachev and V. Iskovskikh.

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- I classified groups of odd order in $\mathrm{Cr}_2(\mathbb{R})$ and groups which are regularized on real del Pezzo surfaces (2016, 2019).
- However, the case of involutions in $\mathrm{Cr}_2(\mathbb{R})$ is very subtle.

  Our first guess was that classification should look similar to the Bayle-Beauville's classification.

  However, this is totally false!

Some remarks

- The classification of involutions in $\mathrm{Cr}_2(\mathbb{R})$ is much longer (than over $\mathbb{C}$) and there are more «types».

- One of the most important issues is that

> the isomorphism type of the fixed curve does not determine the conjugacy class on an involution!

Let me illustrate this phenomena on a very interesting (new) class of involutions in $\mathrm{Cr}_2(\mathbb{R})$.

Kowalevski involutions

Let S be a **complex** del Pezzo surface of degree 2 over a field $\mathbb{K}$ (say, of characteristic zero). Recall that S is the double cover

$$\pi : S \to \mathbb{P}^2_{\mathbb{K}},$$

ramified along the smooth quartic curve $B \subset \mathbb{P}^2_{\mathbb{K}}$. The Galois involution of this double cover is the Geiser involution.

Now let $G \subset \mathrm{Aut}(S)$ be a group of order 2, generated by an involution τ.

- Assume $\mathbb{K} = \mathbb{C}$. It is not difficult to see that the condition $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$ implies that τ is the Geiser involution.

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- Assume $\mathbb{K} = \mathbb{C}$. It is not difficult to see that the condition $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$ implies that τ is the Geiser involution.
- However, if $\mathbb{K} = \mathbb{R}$, there are other involutions which satisfy the condition $\mathrm{Pic}(S)^G = \mathrm{Pic}(S_{\mathbb{C}})^{\Gamma \times G} \simeq \mathbb{Z}$.

Some historical background

In her PhD thesis, Sophie Kowalevski proved the following

Theorem (S. Kowalevski, 1884): Let B be a smooth quartic curve in $\mathbb{P}^2$. Then four bitangents of the curve B meet at one point q if and only if there exists a biregular involution $\kappa \in \mathrm{Aut}(\mathbb{P}^2)$ which leaves B invariant and has the point $q \notin B$ as an isolated fixed point.

THEOREM 1.1 (Kovalevskaia [K]). *Let $y = f(x)$ be an algebraic function satisfying a quartic equation $F(x, y) = 0$. Then there exists an abelian integral of the first kind*

$$u = \int \Psi(x, f(x))\, dx$$

that can be reduced to an elliptic integral using a change of variables of degree 2 if and only if the quartic curve C defined by $F(x, y) = 0$ has four double tangent lines meeting at one point.

«Twenty-eight Double Tangent Lines of a Plane Quartic Curve with an Involution and the Mordell-Weil Lattices» by M. Kuwata

Let me call such a pair (B, κ) the Kowalevski pair. We can assume that our κ acts by $[x:y:z] \mapsto [x:-y:z]$. Then the equation $f_4(x,y,z)=0$ of B takes a special form and we are able to prove

Theorem [CMYZ]: Suppose $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$, the involution τ is not the Geiser involution, and $F(\tau) \neq \varnothing$. Then one can choose coordinates on $\mathbb{P}^2_{\mathbb{R}}$ and $\mathbb{P}_{\mathbb{R}}(2,1,1,1) \supset S = \{w^2 = f_4(x,y,z)\}$ such that the involution τ is given by

$$[x:y:z:w] \mapsto [x:-y:z:w],$$

i.e. it is a positive lift of κ. Moreover, $B(\mathbb{R})$ consists of either one oval, or two nested ovals in $\mathbb{P}^2_{\mathbb{R}}(\mathbb{R})$. In the first case, one has $S(\mathbb{R}) \approx \mathbb{S}^2$ and in the second case one has $S(\mathbb{R}) \approx \mathbb{S}^1 \times \mathbb{S}^1$.

In both cases, $F(\tau)$ is an elliptic curve.

Concluding remarks

- We called these two types of involutions spherical Kowalevski involutions and toroidal Kowalevski involutions.
- As I mentioned, they both fix an elliptic curve.
- However, using the Sarkisov theory, one can easily show that these involutions are not conjugate to each other.

Thank you!