A tale of two widths: lattice + Gromov.

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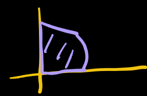
(joint w/ Julian Chaidez)

Basic object

$\Omega \subseteq \mathbb{R}^n$ convex

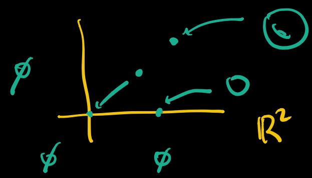
- i) Ω polytope — moment polytopes
- ii) $\partial\Omega$ rat'l-sloped (i.e. facet normals are rat'l)

- a) $\Omega \subseteq \mathbb{R}^n_{\geq 0}$ — convex domain
- b)  (up to $\mathrm{Aff}_n(\mathbb{Z})$ this means I have a smooth vertex)

Geometry

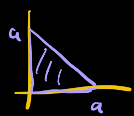
$$(S^1)^n \curvearrowright \mathbb{C}^n \xrightarrow{\mu} \mathbb{R}^n = \mathrm{Lie}((S^1)^n), \quad (z_i) \longmapsto (\pi |z_i|^2).$$

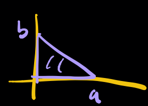
Ham. group action — symplectic $2n$-mfd

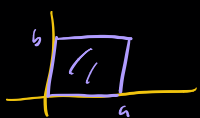


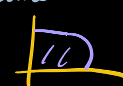
Defn. Let $\Omega \subseteq \mathbb{R}^n_{\geq 0}$ be a convex domain. Then $X_\Omega := \mu^{-1}(\Omega)$ is a <u>convex toric domain</u>.

Convex toric domains are toric symplectic $2n$-manifolds.

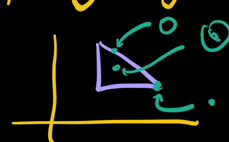
e.g. i) Ω:  $X_\Omega = B^{2n}(a)$

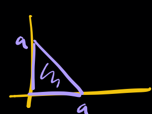
ii) Ω:  $X_\Omega = E(a,b) = \left\{ (z_1,z_2) : \frac{\pi |z_1|^2}{a} + \frac{\pi |z_2|^2}{b} \leq 1 \right\}.$

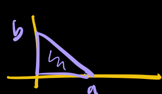
iii)  $X_\Omega = P(a,b)$.

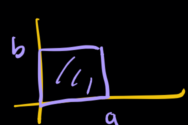
(note: also get some weird ones) 

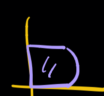
In AG: to a moment polytope Ω one can associate a polarized, possibly singular toric variety (Y_Ω, A_Ω) with moment map $\mu : Y_\Omega \to \mathbb{R}^n$



e.g. i) Ω:  $(Y_\Omega, A_\Omega) = (\mathbb{P}^2, \mathcal{O}(a))$

ii) Ω:  $(Y_\Omega, A_\Omega) = (\mathbb{P}(1,a,b), \mathcal{O}(d))$

iii) Ω:  $(Y_\Omega, A_\Omega) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a,b))$

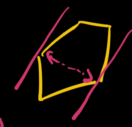


$$\boxed{\text{Key observation: } X_\Omega^\circ = Y_\Omega \setminus A_\Omega \qquad \text{e.g. } B^4(a)^\circ = \mathbb{P}^2 \setminus aH}$$

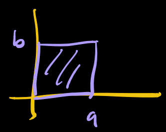
symplectic well-studied — algebraic object

Two widths

Lattice width: $\Omega \subseteq \mathbb{R}^n$ (polytope) define $\ell w(\Omega) := \min\limits_{\ell \in \mathbb{Z}^n \setminus 0} \ \max\limits_{p,q \in \Omega} \langle \ell, p-q \rangle$



e.g.

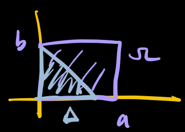


$\min\limits_{(\ell_1,\ell_2)} \ \max \ \ell_1(p_1-q_1) + \ell_2(p_2-q_2) = \min\limits_{(\ell_1,\ell_2)} a\ell_1 + b\ell_2 = \min\{a,b\}.$

Gromov width: Let X be a symplectic $2n$-mfd. Define $c_G(X) := \sup\{a>0 : B^{2n}(a) \overset{s}{\hookrightarrow} X\}$

smooth embedding respects symp. str.

e.g.



assume $b \le a$.

$X_\Delta = B^4(b) \overset{s}{\hookrightarrow} X_\Omega = P(a,b)$

$\Rightarrow \quad c_G(P(a,b)) \ge b = \min\{a,b\} = \ell w(\Omega).$

$\Rightarrow \quad c_G(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a,b)) \ge b.$

Conjecture (Averkov – Hofscheier – Nill '19). Let Ω be a moment polytope. Then
$c_G(Y_\Omega) \le \ell w(\Omega)$.

Key step required: $c_G(Y_\Omega) \le c_G(Y_\Delta)$ if $\Omega \subseteq \Delta$.

"Gromov monotonicity"

Thm. (Chaidez - W. '20) This conjecture is true in dimension 2 (at least when Ω has one smooth vertex).

Thm. (Chaidez - W. '20) $\Omega \subseteq \Delta$ as above then $c_G(Y_\Omega) \le c_G(Y_\Delta)$.

Strategy.
- prove Gromov monotonicity using embedding obstructions in symplectic geometry with nice avatars in AG.
- reduce to $\mathbb{P}^1 \times \mathbb{P}^1$ and apply Gromov nonsqueezing. ← won't talk about.

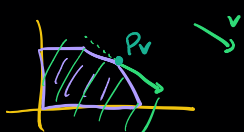
Gromov monotonicity.

I'll introduce some invariants:

V.1. X symplectic 4-mfd $\rightsquigarrow$ $c_k(X)$ = min area of a set of 'Reeb orbits' w/ index $= k$.
"ECH capacities" (Hutchings)

Thm. $X \overset{s}{\hookrightarrow} X'$, $c_k(X) \le c_k(X')$ $\forall k \in \mathbb{Z}_{\ge 0}$. "embedding obstructions"

V.2.



$\|v\|_\Omega = \det(v \mid p_v)$.
"Ω-norm"

$c_k(\Omega) = \min\limits_{\substack{\text{convex lattice} \\ \text{paths}}} \{\Omega\text{-norm of path} : \text{path encloses } k+1 \text{ lattice points}\}$

V.3. (Y, A) polarized surface, define

$$c_k^{alg}(Y,A) := \min_{D \in Nef(Y)_{\mathbb{Z}}} \{ D\cdot A \;:\; \chi(D) \geq k + \chi(\mathcal{O}_Y)\}.$$

If $(Y,A) = (Y_\Omega, A_\Omega)$ then $c_k^{alg}(Y_\Omega, A_\Omega) = \min_{Nef(Y)_{\mathbb{Z}}} \{ D\cdot A_\Omega : h^0(D) \geq k+1 \}$.

Thm. (CCFHR, W.) These three invariants agree when Ω is a rational sloped polygon that is also a convex domain.

Main point: $c_k(X_\Omega)$ encoding embeddings into X_Ω
$\qquad\quad\; \parallel$
$c_k^{alg}(Y_\Omega, A_\Omega)$ encoding AG information.

Thm. (Chaidez – W.) $B(a)^\circ \hookrightarrow Y_\Omega \iff c_k(B(a)) \leq c_k^{alg}(Y_\Omega, A_\Omega)$.

Follows from a result Dan CG: $B(a)^\circ \hookrightarrow X_\Omega \iff c_k(B(a)) \leq c_k(X_\Omega)$.

Proof: $B(a)^\circ \hookrightarrow Y_\Omega \; \overset{\text{SW theory}}{\underset{\text{birational}}{\rightsquigarrow}} \; c_k(B(a)) \leq c_k^{alg}(Y_\Omega, A_\Omega)$.

Suppose $c_k(B(a)) \leq c_k^{alg}(Y_\Omega, A_\Omega) \Rightarrow c_k(B(a)) \leq c_k(X_\Omega)$

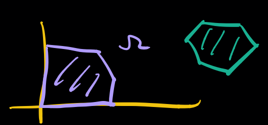
$\overset{CG}{\Rightarrow} B(a)^\circ \hookrightarrow X_\Omega^\circ$

$\Rightarrow B(a)^\circ \hookrightarrow Y_\Omega$ $\square$

Cor. $\Omega \subseteq \Delta$ w/ one smooth vertex $\Rightarrow c_G(X_\Omega) \leq c_G(X_\Delta)$.

Proof: Just need that $c_k^{alg}(Y_\Omega, A_\Omega) \leq c_k^{alg}(Y_\Delta, A_\Delta)$. This follows from $c_k^{alg}(Y_\Omega) = c_k(X_\Omega)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wedge\!\mid$
$c_k^{alg}(Y_\Delta) = c_k(X_\Delta)$. $\square$

Why the smooth fixed pt / how to get rid of it?



$c_k^{alg}(Y, A) = c_k^{alg}(\tilde{Y}, \pi^* A)$, $\pi : \tilde{Y} \to Y$ blowup.