Mirror symmetry for parabolic Higgs bundles, from Local to Global

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Outline

History and Motivation

1. Local Mirror Symmetry
   - Nilpotent orbits
   - Seesaw phenomenon and the footprint for Richardson orbits
   - Mirror symmetry for parabolic covers of Richardson orbits
2. Global Mirror Symmetry
   - SYZ and Topological Mirror Symmetries

Based on:

1. joint work with B. Fu and Y. Ruan, arXiv:2207.10533
2. in-progress work with W. He, X. Su, B. Wang, and X. Wen

History

[Hitchin, 86] studied the space of special solutions of the self-dual equations.

$$\text{4d super Yang-Mills theory} \overset{\textit{reduction}}{\rightsquigarrow} \text{Hitchin's equations}$$

$$F_A - \phi \wedge \phi = 0$$
$$d_A\phi = 0, d_A * \phi = 0$$

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It admits a hyperKähler structure. Furthermore

$$\begin{aligned} SL_r - Higgs^s(C,d) &\xrightarrow{h} \mathcal{A} \\ (E,\phi) &\mapsto \det(\lambda - \phi) \end{aligned}$$

the Hitchin map is projective which makes the moduli space a completely integrable system.

$$\text{Hitchin base:} \quad \mathcal{A} = \bigoplus_{i=2}^{r} H^0(C, K_C^i).$$

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In [Hausel-Thaddeus, 02], they proposed two kinds of mirror symmetries:

SYZ mirror symmetry and Topological mirror symmetry

SYZ Mirror Symmetry

[Hausel-Thaddeus, 02] SYZ:

$$SL_r - Higgs^s(C,d) \xrightarrow{h} \mathcal{A} \xleftarrow{{}^L h} PGL_r - Higgs^s(C,d)$$

For generic $a \in \mathcal{A}$,

$$h^{-1}(a), \quad {}^L h^{-1}(a)$$

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- HyperKähler structure $\Rightarrow$ special Lagrangian
- BNR correspondence $\Rightarrow$ abelian variety of the Hitchin fiber

Topological Mirror Symmetry

Stringy E-functional: Let M be a normal variety with only canonical singularities. Consider a log resolution

$$\rho : Z \longrightarrow M,$$

i.e., the exceptional locus of ρ is a divisor whose irreducible components $D_1, \cdots, D_s$ are smooth with only normal crossing. And

$$K_Z = \rho^* K_M + \sum_{i=1}^{s} a_i D_i, \quad a_i \geq 0.$$

Topological Mirror Symmetry

For any subset $J \subseteq I = \{1, \cdots, s\}$, let

$$D_J = \bigcap_{j \in J} D_j, \quad D_J^\circ = D_J - \bigcup_{i \in I \setminus J} D_i.$$

Then the stringy E-functional of M is defined by

$$E_{st}(M; u, v) = \sum_{J \subseteq I} E(D_J^\circ; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j + 1} - 1},$$

where $E(D_J^\circ; u, v)$ is the Hodge-Deligne polynomial

$$E(D_J^\circ; u, v) = \sum_{p,q} \sum_{k \geq 0} (-1)^k h^{p,q}(H_c^k(D_J^\circ; \mathbb{C})) u^p v^q.$$

It is well-known by [Batyrev, 97] that the stringy E-functional is independent of the choice of the resolution.

Topological Mirror Symmetry

[Hausel-Thaddeus, 02] TMS:

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- [Groechenig-Wyss-Ziegler, 17] via p-adic integration.
- [Maulik-Shen, 20] via support theorem and vanishing cycle techniques.

Motivation: Geometric Langlands and surface operator

[Kapustin-Witten, 06] initiated a program to study Langlands program via 4d gauge theory and S-duality. [Gukov-Witten, 06] ([Gukov-Witten, 08]) introduced (rigid) surface operators in gauge theory.

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$$\begin{array}{l} F_A - \phi \wedge \phi = 0 \\ d_A\phi = 0, d_A * \phi = 0 \\ \text{with singularities} \end{array} \overset{\text{step 1}}{\rightsquigarrow} \begin{array}{l} \frac{\mathrm{d}a}{s} = [b, c] \\ \frac{\mathrm{d}b}{s} = [c, a] \\ \frac{\mathrm{d}c}{s} = [a, b] \end{array} \overset{\text{step 2}}{\rightsquigarrow} \text{Coadjoint orbit}$$

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Nahm's equations were first used by [Kronheimer, 89] to construct the hyperKähler structure on coadjoint orbits of a certain type. It was generalized to any type by [Kovalev, 94].

Parabolic Higgs bundle

Hitchin's equations with singularities were first studied by [Simpson, 90]. For type A, fix a point $x \in C$, and filtration of bundle $F^\bullet(E_x)$ at x, i.e.,

$$F^\bullet(E_x) : E_x = E_0 \supset E_1 \supset \cdots \supset E_{d-1} \supset E_d = 0$$

$$PHiggs(C, r, d, F^\bullet(E_x)) = \left\{ (E, \phi) \mid \begin{array}{l} \phi : E \to E \otimes K_C(x) \\ \text{Res}_x\, \phi \quad \text{preserves} \quad F^\bullet(E_x) \end{array} \right\} /$$

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- Weakly preserve: $\mathrm{Res}_x(\phi)(E_i) \subset E_i$,
- Strongly preserve: $\mathrm{Res}_x(\phi)(E_i) \subset E_{i+1}$.

SYZ and TMS for type A

SYZ: [Su-Wang-X.Wen, 19] via parabolic BNR correspondence.

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Remark

In type A, there is a one-to-one correspondence between classes of filtrations and nilpotent orbits.

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Local mirror symmetry for nilpotent orbit closure

Let G be a complex semisimple Lie group of classical type, and $\mathfrak{g}$ be its Lie algebra. Let $X \in \mathfrak{g}$ be a nilpotent element, denote by

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It is known that [Borel, Harish-Chandra] $\mathbf{O}_X$ is closed if and only if X is semisimple. Then nilpotent orbit is not closed in $\mathfrak{g}$.

Nilpotent orbits

We say $\mathbf{d} = [d_1, d_2, \cdots] \geq \mathbf{f} = [f_1, f_2, \cdots]$ if $\sum_{i=1}^k d_i \geq \sum_{i=1}^k f_i$ for any $k \geq 1$. Then

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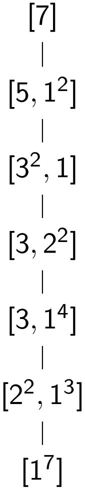

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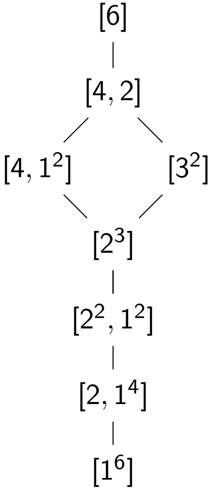

Special nilpotent orbits

If the transpose of the partition $\mathbf{d}$, denote by $\mathbf{d^t}$, is still the same type. We call the associate nilpotent orbit $\mathbf{O_d}$ *special*.

For examples:

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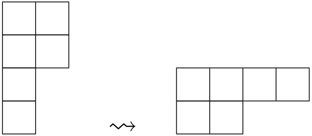

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  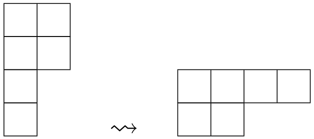

  $\mathbf{d}^{\mathbf{t}} = [\mathbf{4}, \mathbf{2}]$ is still of type C. Thus $[2^2, 1^2]$ is special.
- $\mathbf{d} = [\mathbf{2}, \mathbf{1}^4]$. $\mathbf{d}^{\mathbf{t}} = [\mathbf{5}, \mathbf{1}]$ is not of type C. Then $[2, 1^4]$ is not special.

Special orbits

$\mathfrak{so}_7$:

$[7]$

$[5, 1^2]$

$[3^2, 1]$

$[3, 2^2]$

$[3, 1^4]$

$[2^2, 1^3]$ (red)

$[1^7]$

$\mathfrak{sp}_6$:

$[6]$

$[4, 2]$

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$[2^3]$

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$[2, 1^4]$ (red)

$[1^6]$

Special nilpotent orbits

Denote by $\mathcal{N}^{\mathrm{sp}}$ the set of special orbits. Then Springer theorem gives a one-to-one correspondence of special orbits in Lie algebra of type B_n and C_n:

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$\mathfrak{so}_7$:

$[3, 1^4]$, dim=10
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$[2^2, 1^3]$, dim=8
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$[1^7]$, dim=0

$\mathfrak{sp}_6$:

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Question: How to remedy the failure?

Richardson orbits

We say a nilpotent orbit **O** is *Richardson* if there exists a parabolic subgroup $P < G$ such that

$$\mu_P : T^*(G/P) \twoheadrightarrow \overline{\mathbf{O}}.$$

We call P a *polarization* of **O** and $\mathrm{Pol}(\mathbf{O})$ the set of classes of all polarizations of the orbit.

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The Springer map μ_P is generically finite. If $\deg(\mu_P) = 1$, then it is *crepant*.

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The Springer map μ_P is generically finite. If $\deg(\mu_P) = 1$, then it is *crepant*. Conversely, if $\overline{\mathbf{O}}$ admits a crepant resolution, i.e.,

$$\rho : Z \to \overline{\mathbf{O}}.$$

Then $\mathbf{O}$ is Richardson and $Z \cong T^*(G/P)$ for some $P < G$ (by [Fu, 03]).

Failure of the naive thought

$\mathbf{O}_{[3,1^4]}$ and $\mathbf{O}_{[2^2,1^2]}$ are Richardson:

$$
\begin{array}{ccc}
T^*(G/P) & \overset{\textit{Langlands dual}}{\rightsquigarrow} & T^*({}^L G/{}^L P)\ . \\
\Big\downarrow {\color{blue}1:1} & & \Big\downarrow {\color{red}2:1} \\
\overline{\mathbf{O}}_{[3,1^4]} & \overset{\textit{Springer dual}}{\rightsquigarrow} & \overline{\mathbf{O}}_{[2^2,1^2]}
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$$\begin{array}{ccc} T^*(G/P) & \overset{Langlands\ \ dual}{\rightsquigarrow} & T^*({}^LG/{}^LP)\,. \\ \color{blue}{1{:}1}\Big\downarrow & & \Big\downarrow\color{red}{2{:}1} \\ \overline{\mathbf{O}}_{[3,1^4]} & \overset{Springer\ \ dual}{\rightsquigarrow} & \overline{\mathbf{O}}_{[2^2,1^2]} \end{array}$$

$$\begin{aligned} E_{st}(\overline{\mathbf{O}}_{[3,1^4]}) &= E(T^*(G/P)) = E(G/P)q^5, \quad q = uv \\ &= q^5(1 + q + q^2 + q^3 + q^4 + q^5) \\ E_{st}(\overline{\mathbf{O}}_{[2^2,1^2]}) &= \frac{(q^4-1)(q^5-1)(q^6-1)q^3}{(q^2-1)(q^3-1)(q^3-1)}. \end{aligned}$$

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However, by a little computation, one finds that

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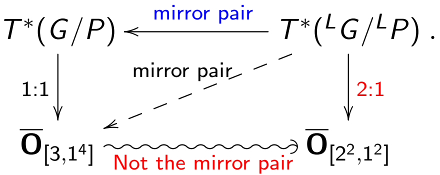

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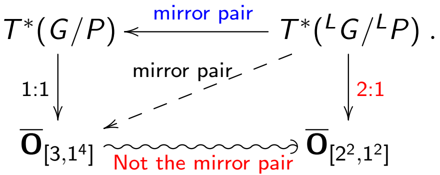

A way to remedy the failure: consider certain cover of the nilpotent orbit closure!

Seesaw phenomenon

Let $P < G$ be a parabolic subgroup with Lie algebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ and ${}^L P < {}^L G$ the Langlands dual parabolic subgroup with Lie algebra ${}^L\mathfrak{p} = {}^L\mathfrak{l} \oplus {}^L\mathfrak{u}$.

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Theorem ([Fu-Ruan-Wen, 22])

$$\begin{array}{ccc} T^*(G/P) & \overset{Langlands\ \ dual}{\rightsquigarrow} & T^*({}^LG/{}^LP)\ . \\ \Big\downarrow{\scriptstyle \mu_P} & & \Big\downarrow{\scriptstyle \mu_{{}^LP}} \\ \overline{\mathbf{O}} & \overset{Springer\ \ dual}{\rightsquigarrow} & \overline{{}^S\mathbf{O}} \end{array}$$

Moreover, we have the following seesaw property for the degrees:

$$\deg \mu_P \cdot \deg \mu_{{}^LP} = |\bar{A}(\mathbf{O})| = |\bar{A}({}^S\mathbf{O})|.$$

Mirror symmetry for Richardson orbits

Taking Stein factorization, we have

$$\begin{array}{ccccccc} & & T^*(G/P) & \overset{Langlands\ dual}{\rightsquigarrow} & T^*({}^LG/{}^LP) & & \\ & \swarrow^{\pi_P} & \downarrow \mu_P & & \downarrow \mu_{{}^LP} & \searrow^{\pi_{{}^LP}} & \\ X_P & & & & & & X_{{}^LP} \\ & \searrow_{\nu_P} & \downarrow & & \downarrow & \swarrow_{\nu_{{}^LP}} & \\ & & \widetilde{\mathbf{O}} & \overset{Springer\ dual}{\rightsquigarrow} & \widetilde{{}^S\mathbf{O}} & & \end{array}$$

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\end{array}
$$

$$\overset{Springer\ \ dual}{\rightsquigarrow} \quad (\widetilde{\mathbf{O}} \rightsquigarrow \widetilde{{}^S\mathbf{O}})$$

where π_P (resp. $\pi_{{}^LP}$) is birational, and ν_P (resp. $\nu_{{}^LP}$) is a finite map. We call X_P (resp. $X_{{}^LP}$) the *parabolic cover* of $\widetilde{\mathbf{O}}$ (resp. $\widetilde{{}^S\mathbf{O}}$) associated with P (resp. ${}^LP$), which is normal with only canonical singularities.

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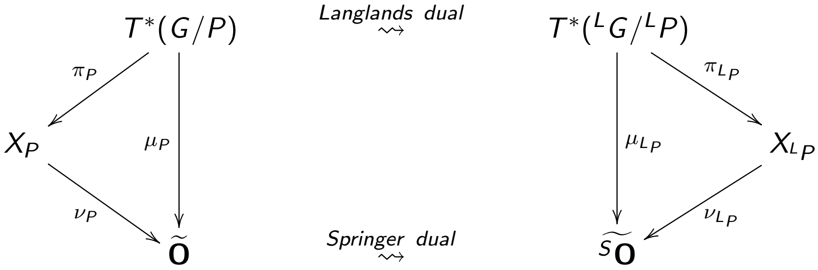

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Mirror symmetry for Richardson orbits

Proposition (Topological mirror symmetry,[Fu-Ruan-Wen, 22])

For any polarization P of a Richardson orbit $\mathbf{O}$, the two Springer dual parabolic covers X_P and $X_{^LP}$ share the same stringy E-polynomial.

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Proposition ([Fu-Ruan-Wen, 22])

Given a Springer dual pair $(\mathbf{O}, {}^S\mathbf{O})$ of Richardson orbits, we have

$$\begin{aligned}&\{(\deg \mu_P, \deg \mu_{^LP}) | P \in \mathrm{Pol}(\mathbf{O})\} \\ =&\{(2^\beta, 2^{\alpha+m}), (2^{\beta+1}, 2^{\alpha+m-1}), \cdots, (2^{\beta+m}, 2^\alpha)\}.\end{aligned}$$

We call the set $\{(\deg \mu_P, \deg \mu_{^LP})\}$ the *footprint*.

Asymmetry for the footprint

Consider $[3,1^4] \in \mathfrak{so}_7$ and $[2^2,1^2] \in \mathfrak{sp}_6$. $\bar{A}(\mathbf{O}) = \mathbb{Z}_2$.

Mirror Pair	$[3,1^4]$	$[2^2,1^2]$
All Polarizations	P	${}^LP$
Degree of Springer map	1	2

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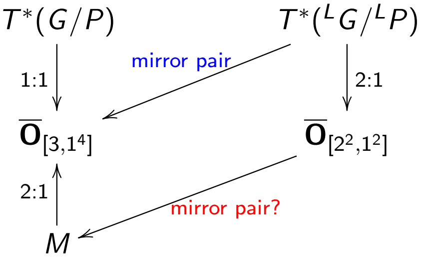
Question: What happens if we go beyond the range of the footprint?
How about $(2, 1)$?

Asymmetry for the footprint

Since $\pi_1(\mathbf{O}_{[3,1^4]}) = \bar{A}(\mathbf{O}_{[3,1^4]}) = \mathbb{Z}_2$. Let M be the double cover of $\overline{\mathbf{O}}_{[3,1^4]}$, then



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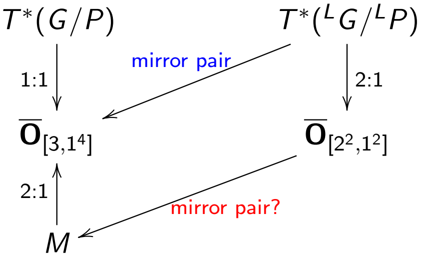

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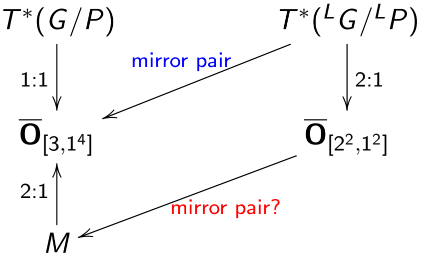

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In the following, for simplicity, we denote $\mathbf{O}_{[3,1^4]}$ and $\mathbf{O}_{[2^2,1^2]}$ by $\mathbf{O}_B$ and $\mathbf{O}_C$ respectively.

Asymmetry for the footprint

Questions: 1. What is the M?

Asymmetry for the footprint

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Proposition ([Fu-Ruan-Wen, 22])

Consider the following nilpotent orbit

$$\mathbf{O}_D := \mathbf{O}_{[2^2,1^4]} \subset \mathfrak{so}_8.$$

Then there exists an SO_7*–equivariant double cover* $\overline{\mathbf{O}}_D \to \overline{\mathbf{O}}_B$.

2. How to compute $E_{st}(\overline{\mathbf{O}}_D)$ and $E_{st}(\overline{\mathbf{O}}_C)$? Are they the same?

Log resolution of orbit closures

There are so-called Jacobson-Morosov resolutions for $\overline{\mathbf{O}}_C$ and $\overline{\mathbf{O}}_D$:

$$G_C \times_{P_C} \mathfrak{n}_C \longrightarrow \overline{\mathbf{O}}_C, \quad G_D \times_{P_D} \mathfrak{n}_D \longrightarrow \overline{\mathbf{O}}_D.$$

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Proposition ([Fu-Ruan-Wen, 22])

Under the action of P_C (resp. P_D), $\mathfrak{n}_C$ (resp. $\mathfrak{n}_D$) becomes an SL_{2r}–module. Moreover, there exist two vector spaces $V_C \simeq V_D \simeq \mathbb{C}^{2r}$ such that (in previous example $r = 1$)

$$\mathfrak{n}_C = \mathrm{Sym}^2 V_C \quad \mathfrak{n}_D = \wedge^2 V_D.$$

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The Jacobson-Morosov resolution is generally not a log resolution, but we will construct a log resolution from it by successive blowups.

Log resolution of type C

- Let $M_k \subset \mathrm{Sym}^2 V_C$ be the set of elements of rank k, $k = 0, 1, 2$.
- $G_C \times_{P_C} M_k \longrightarrow \mathbf{O}_k^C := \mathbf{O}_{[2^k, 1^{6-2k}]} \subset \overline{\mathbf{O}}_C = \bigsqcup_{i=0}^2 \mathbf{O}_i^C$.

Consider the following birational map

$$\phi : \widehat{\mathfrak{n}}_C \to \mathfrak{n}_C = \mathrm{Sym}^2 V_C$$

obtained by successive blowups of $\mathfrak{n}_C = \mathrm{Sym}^2 V_C$ along strict transforms of $\overline{M}_i$ from smallest M_0 to the biggest $\overline{M}_{2r-2}$.

Finally, we have the following log resolution

$$\Phi : \widehat{Z}_C := G_C \times_{P_C} \widehat{\mathfrak{n}}_C \to Z_C := G_C \times_{P_C} \mathfrak{n}_C \to \overline{\mathbf{O}}_C.$$

Log resolution of type C

$$E_{st}(M;u,v)=\sum_{J\subseteq I}E(D_J^\circ;u,v)\prod_{j\in J}\frac{uv-1}{(uv)^{a_j+1}-1},$$

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Let us denote by $\mathcal{D}_i^C$ the exceptional divisor of Φ over $\overline{\mathbf{O}}_i^C$ for $i=0,\cdots,2r-1$.

Proposition ([Fu-Ruan-Wen, 22])

The morphism Φ is a log resolution for $\overline{\mathbf{O}}_{r,l}^C$, and we have

$$K_{\widehat{Z}_C}=2l\mathcal{D}_{2r-1}^C+\sum_{j=0}^{2r-2}\left(\frac{(2r-j)(2r+1-j)}{2}-1\right)\mathcal{D}_j^C.$$

In the previous example, $r=l=1$.

compare $E_{st}(\overline{\mathbf{O}}_D)$ and $E_{st}(\overline{\mathbf{O}}_C)$

In our previous example

$$\mathbf{O}_B = \mathbf{O}_{[3,1^4]} \subset \mathfrak{so}_7, \quad \mathbf{O}_C = \mathbf{O}_{[2^2,1^2]} \subset \mathfrak{sp}_6,$$
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$$\mathrm{E}_{\mathrm{st}}(\overline{\mathbf{O}}_D) = \frac{(q^2+1)(q^4-1)(q^6-1)q^5}{(q^2-1)(q^5-1)}.$$

$$\mathrm{E}_{\mathrm{st}}(\overline{\mathbf{O}}_C) = \frac{(q^4-1)(q^5-1)(q^6-1)q^3}{(q^2-1)(q^3-1)(q^3-1)}.$$

Global Mirror Symmetry

Firstly, we need to construct a moduli space associated with the Jacobson-Morozov resolution of the nilpotent orbit closure, i.e.,

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Here $\mathcal{PA} = \bigoplus_{i=2}^{n} H^0(C, K_C^{2i}((2i - \delta_i)x))$, here $\{\delta_i\}$ is called the *singularity* of the spectral curve.

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Here $\mathcal{PA} = \bigoplus_{i=2}^{n} H^0(C, K_C^{2i}((2i - \delta_i)x))$, here $\{\delta_i\}$ is called the *singularity* of the spectral curve.
The Hitchin maps may NOT be surjective in general. If the orbit $\mathbf{O}$ is *special*, then Hitchin map is surjective and proper.

Here the δ_i's are defined as follows. Consider a Richardson orbit of type C

$$\mathbf{O}_{[5,5,4,2]} \subset \mathfrak{sp}_{16}.$$

$$\begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\\hline 2&2&2&2&2\\\hline 3&3&3&3\\\cline{1-4} 4&4\\\cline{1-2}\end{array} \quad \rightsquigarrow \quad \begin{array}{|c|c|c|c|c|}\hline 1&*&1&*&1\\\hline *&2&*&2&*\\\hline 3&*&3&*\\\cline{1-4} 4&*\\\cline{1-2}\end{array}$$

i.e., $\delta_1 = 1$, $\delta_2 = 1$, $\delta_3 = 2$, $\delta_4 = 2$, $\delta_5 = 2$, $\delta_6 = 3$, $\delta_7 = 3$, $\delta_8 = 4$.

Let's consider the Springer dual Richardson orbit of type B:

$$\mathbf{O}_{[5,5,5,1,1]} \subset \mathfrak{so}_{17}.$$

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 & 3 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \quad \rightsquigarrow \quad \begin{array}{|c|c|c|c|c|}\hline 1 & * & 1 & * & 1 \\ \hline * & 2 & * & 2 & * \\ \hline 3 & * & 3 & * & 3 \\ \hline * \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}$$

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Global Mirror Symmetry

Theorem ([He-Su-Wang-X.Wen-Y.Wen, in preparation])

For two nilpotent orbits $\mathbf{O}_B$ *in type B and* $\mathbf{O}_C$ *in type C. Then* $\mathbf{O}_B$ *and* $\mathbf{O}_C$ *are both special and correspondenced by Springer dual if and only if the following two conditions holds:*

1. $\dim JMH(C, \overline{\mathbf{O}}_B, d) = \dim JMH(C, \overline{\mathbf{O}}_C, d)$,
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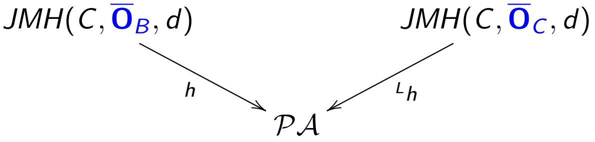

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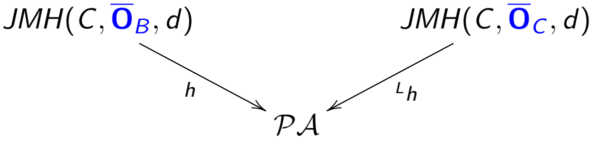

However, the generic Hitchin fibers $h^{-1}(a)$ and ${}^L h^{-1}(a)$ are NOT dual abelian varieties!

SYM for Richardson cases

To remedy this problem, in Richardson cases, we consider

$$
\begin{array}{ccccc}
PHiggs(C, P, d) & & & & PHiggs(C, {}^L P, d) \\
\Big\downarrow{\scriptstyle \mu_P} & \searrow^{h} & & {}^{{}^L h}\swarrow & \Big\downarrow{\scriptstyle \mu_{{}^L P}} \\
JMH(C, \overline{\mathbf{O}}_B, d) & \longrightarrow & \mathcal{PA} & \longleftarrow & JMH(C, \overline{\mathbf{O}}_C, d)
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The maps μ_p and $\mu_{{}^LP}$ between generic fibers are FINITE!

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Theorem ([He-Su-Wang-X.Wen-Y.Wen, in preparation])

For a generic point $a \in \mathcal{PA}$, the generic Hitchin fibers $h^{-1}(a)$ and ${}^L h^{-1}(a)$ are dual abelian varieties.

TMS for Richardson cases

Theorem ([He-Su-Wang-X.Wen-Y.Wen, in preparation])

Two moduli spaces $PHiggs(C, P, d)$ and $PHiggs(C, {}^LP, d)$ with dual input data share the same stringy E-functional.

Via p-adic integration.

Thank you!

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