On The Newton Polytope of the Morse Discriminant of a Univariate Polynomial

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Nottingham Online Algebraic Geometry Seminar

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Example

$A = \{1, 2, 3, 4\} \subset \mathbb{Z}$;
$\mathbb{C}^A = \{b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 \mid b_i \in \mathbb{C}\}$;

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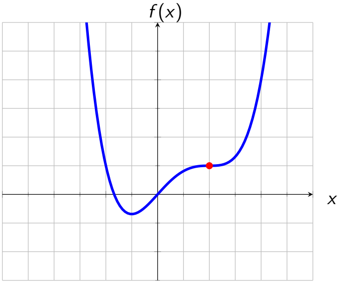
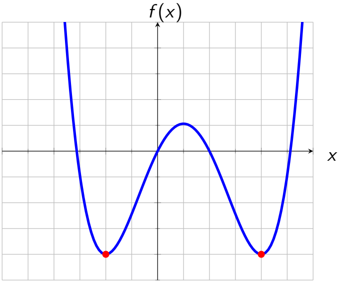


[caustic]

The map $f : (\mathbb{C} \setminus 0) \to \mathbb{C}$ has a degenerate critical point.



[Maxwell stratum]

The map $f : (\mathbb{C} \setminus 0) \to \mathbb{C}$ has a pair of coinciding critical values taken at distinct points.

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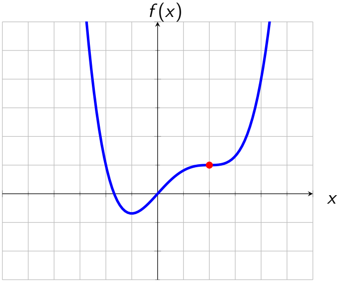

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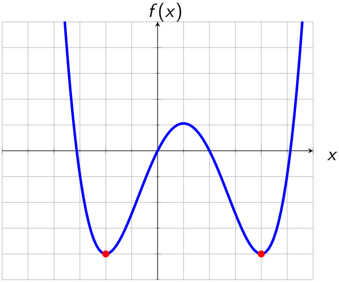

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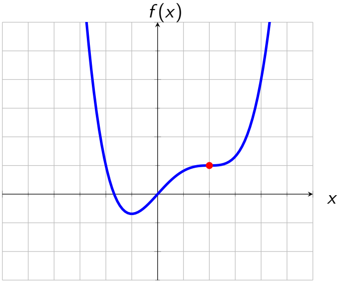
Definition

A polynomial $f \in \mathbb{C}^A$ is *Morse*, if it does not belong to either the caustic or the Maxwell stratum.

Example: $A = \{1, 2, 3, 4\}$



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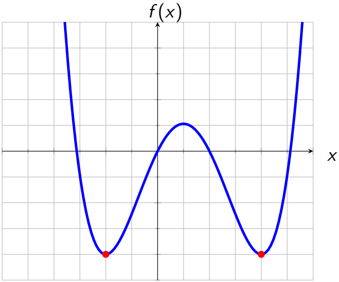

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$\{h_c = 0\} \subset \mathbb{C}^A$ $\qquad$ $\{h_m = 0\} \subset \mathbb{C}^A$

$$h_c = b_2^2 b_3^2 - 4 b_1 b_3^3 - 4 b_2^3 b_4 + 18 b_1 b_2 b_3 b_4 - 27 b_1^2 b_4^2$$

$$h_m = b_3^3 + 8 b_1 b_4^2 - 4 b_2 b_3 b_4$$

Statement of the problem

Problem

Describe in terms of the set A the Newton polytope $\mathcal{M}_A$ of the Morse discriminant, i.e. of the polynomial $h_m^2 h_c$.

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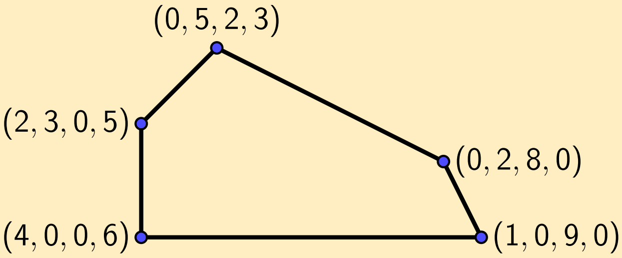
For $A = \{1, 2, 3, 4\}$, the polytope $\mathcal{M}_A$ is a pentagon in $\mathbb{R}^4$.



The support function (reminder)

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Let $P \subset \mathbb{R}^n$ be a convex polytope. Its *support function* $\tilde{P} \colon (\mathbb{R}^n)^* \to \mathbb{R}$ is defined as follows:

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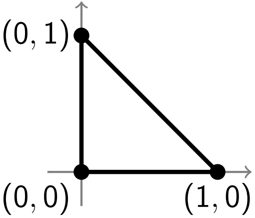

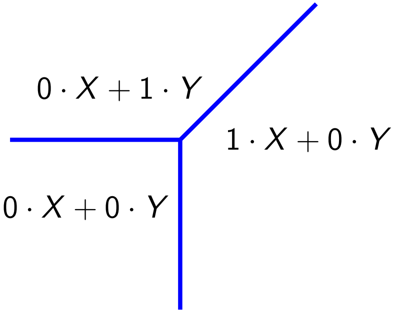

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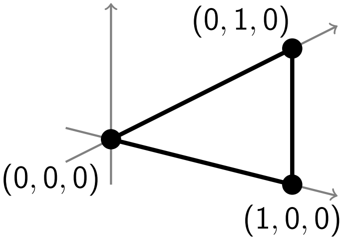

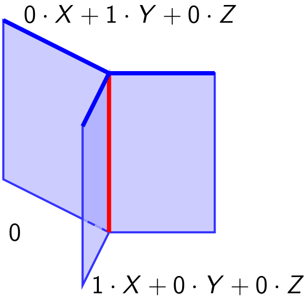

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Tropical semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$:

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$$\gamma = (3, 5, 2, 5, 1) \in (\mathbb{R}^5)^* \leftrightarrow$$
$$\varphi_\gamma(X) = \max(-3X + 3, -X + 5,\ X + 2,\ 2X + 5,\ 4X + 1).$$

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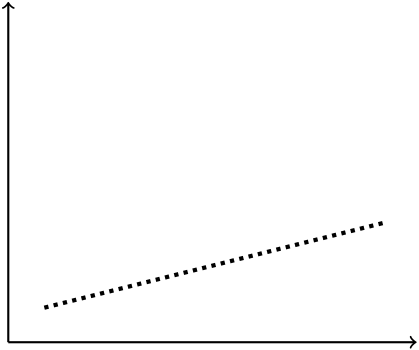

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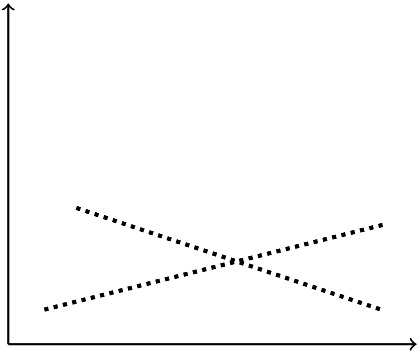

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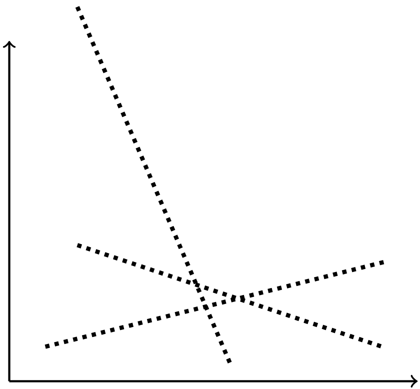

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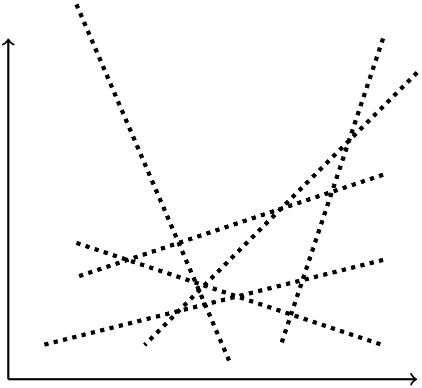

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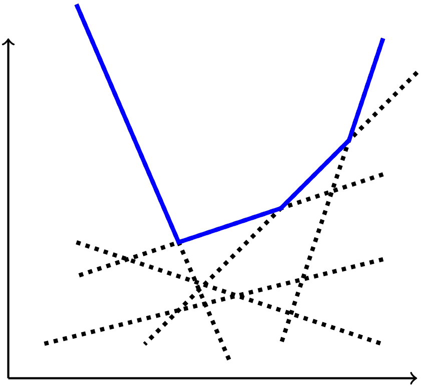

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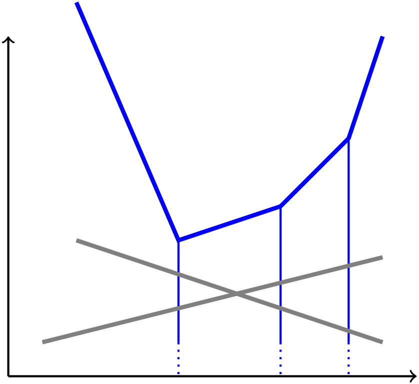

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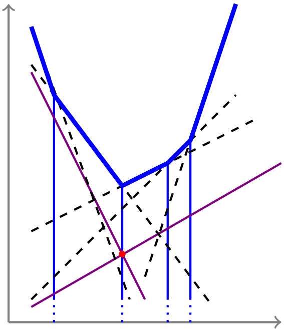

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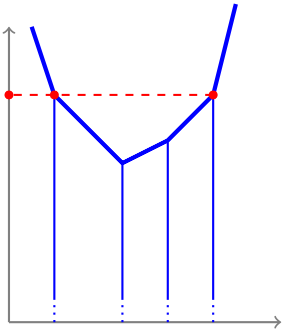

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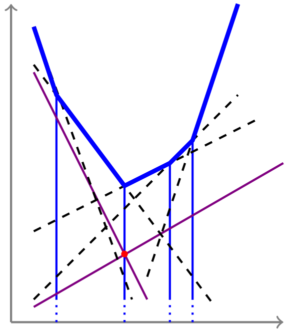

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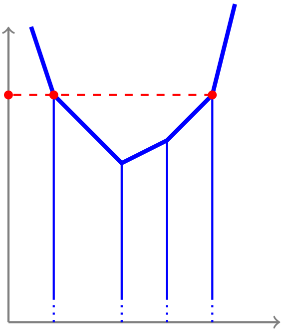

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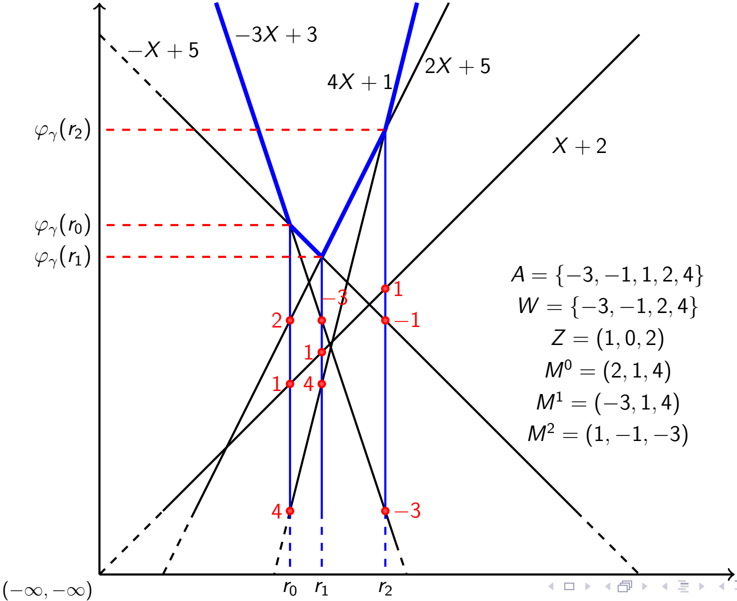
Linearity domain of μ_A $\longrightarrow$ Vertex of $\mathcal{M}_A$

Combinatorial data



$$A = \{-3,-1,1,2,4\}$$
$$W = \{-3,-1,2,4\}$$
$$Z = (1,0,2)$$
$$M^0 = (2,1,4)$$
$$M^1 = (-3,1,4)$$
$$M^2 = (1,-1,-3)$$

Main result

Theorem (A.V.'21)

There is a surjection (given by a certain loooong and scary formula) between the set of all possible combinatorial types of Morse tropical polynomials with support set A and the vertices of the polytope $\mathcal{M}_A$.

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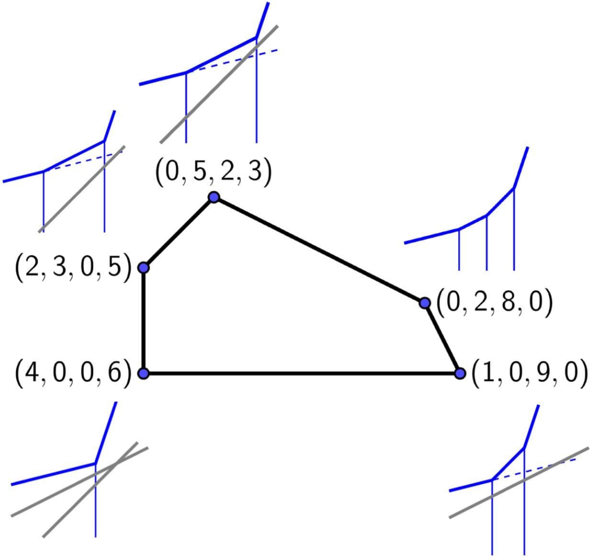
This result allows to enumerate all the vertices of the sought Newton polytope $\mathcal{M}_A$ by all sorts of combinatorial types of Morse tropical polynomials.

Example: $A = \{1, 2, 3, 4\}$



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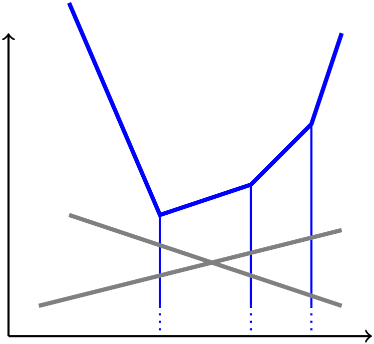
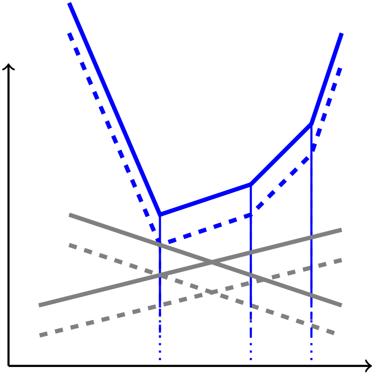


Covectors $\leftrightarrow$ tropical polynomials

$$\gamma' = \gamma + (b, \dots, b);\ b > 0,$$
$$\varphi_{\gamma'}(X) = \bigoplus_{a \in A} \gamma(a) \odot b \odot X^{\odot a} = \max_{a \in A}(aX + \gamma(a) + b).$$

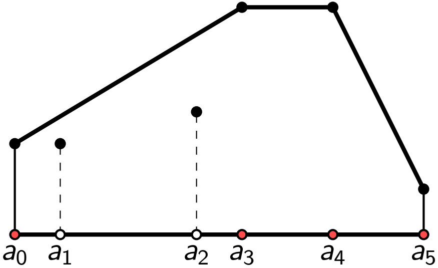


It suffices to consider covectors with non-negative coordinates!

Covectors $\leftrightarrow$ polygons

$$\gamma : A \to \mathbb{R}_{\geqslant 0} \longleftrightarrow N_\gamma \subset \mathbb{R}^2$$

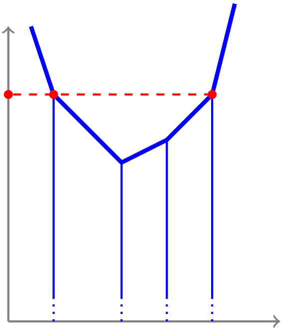
$$N_\gamma = \operatorname{conv}(\{(a, \gamma(a)) \mid a \in A\} \cup \{(a, 0) \mid a \in A\})$$

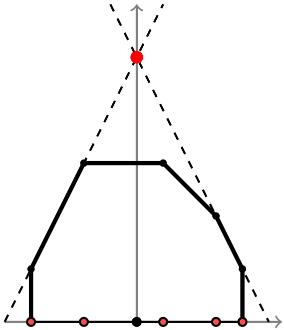


Non-Morse tropical Laurent polynomials revisited

Definition

We say that a tropical Laurent polynomial $F(X)$ belongs to the *tropical Maxwell stratum* in the space of tropical polynomials with the given support A, if there exists a pair r_1, r_2 of tropical roots of $F(X)$, such that $F(r_1) = F(r_2)$.

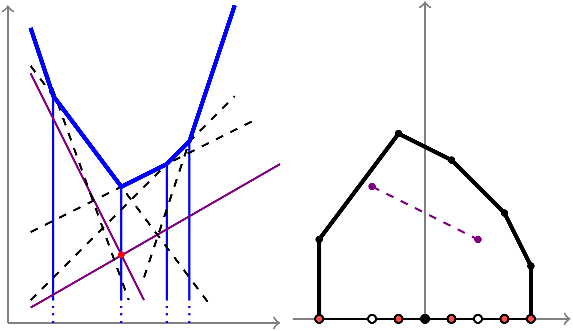




Non-Morse tropical Laurent polynomials revisited

Definition

A tropical Laurent polynomial $F(X)$ belongs to the *tropical caustic* in the space of tropical polynomials with the given support A, if for some tropical root r of $F(X)$, there are at least two pairs of monomials attaining the same values at r.



The support set A

$0 \notin A \subset \mathbb{Z}$, a finite set.

- A affinely generates $\mathbb{Z}$;
- $\mathrm{Length}(\mathrm{conv}\, A) \geqslant 3$;

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True for a wide range of sets $A \subset \mathbb{Z}$. For instance:

- sets A such that $A = \mathrm{conv}(A) \cap \mathbb{Z}$;
- sets A containing 4 consecutive integers.

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Conjecture

Any set A satisfying the first two properties, also satisfies the third one.

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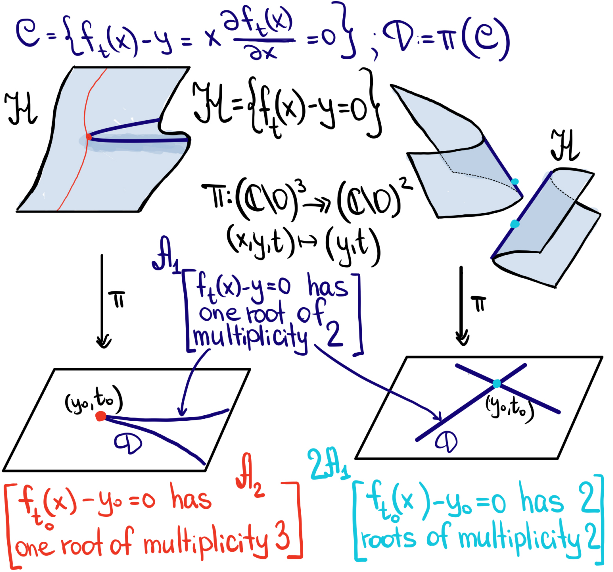
   Thus, we can reformulate the initial problem as follows:

Problem

For how many complex values of t is the polynomial $f_t^\gamma(x)$ non-Morse?

The key ideas



One more statement of the problem

$\mathcal{M}_A \subset \mathbb{R}^{|A|}$ – the Newton polytope of the Morse discriminant,
$\mu_A \colon (\mathbb{R}^{|A|})^* \to \mathbb{R}$ – its support function.

Proposition

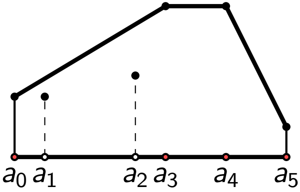
For a generic covector γ with non-negative integer coefficients, we have

$$\mu_A(\gamma) = 2 \cdot \underbrace{|2\mathcal{A}_1|}_{\textit{Maxwell stratum}} + \underbrace{|\mathcal{A}_2|}_{\textit{caustic}}.$$

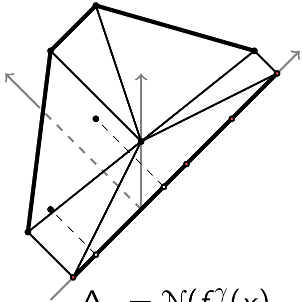
Thus, we reduced the initial problem to finding the number of cusps and nodes of the curve $\mathcal{D}$.

2 polytopes

$\gamma : A \to \mathbb{Z}_{\geqslant 0}$ – a covector;



$N_\gamma = \mathcal{N}(f_t^\gamma(x))$



$\Delta_\gamma = \mathcal{N}(f_t^\gamma(x) - y)$

3 equations

Proposition

$$|\mathcal{A}_2| = \mathrm{Area}(N_\gamma) - \gamma(a_0) - \gamma(a_{|A|-1}).$$

Proof.

Follows from the description of the Newton polytope of the classical discriminant by Gelfand, Kapranov, Zelevinsky. □

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Proposition

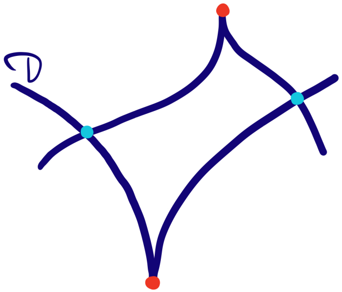
$$\chi(\mathcal{A}_1) + 2|2\mathcal{A}_1| + 2|\mathcal{A}_2| = -\,\mathrm{Area}(N_\gamma)$$

Proof.

Bernstein–Kouchnirenko–Khovanskii theorem + additivity of Euler characteristic. □

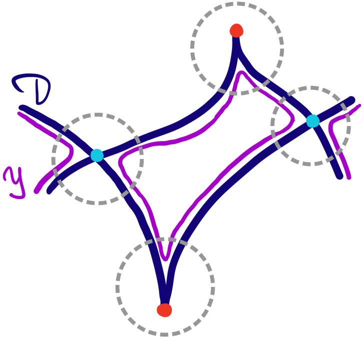
3 equations

The first two equations do not suffice. We need the third one!



$$\chi(\mathcal{D}) = \chi(\mathcal{A}_1) + |2\mathcal{A}_1| + |\mathcal{A}_2|$$

3 equations



$\chi(\mathcal{A}_1) + |2\mathcal{A}_1| + |\mathcal{A}_2| - |2\mathcal{A}_1| - |\mathcal{A}_2| + |2\mathcal{A}_1| \cdot 0 + |\mathcal{A}_2| \cdot (-1) = \chi(\mathcal{A}_1) - |\mathcal{A}_2|$

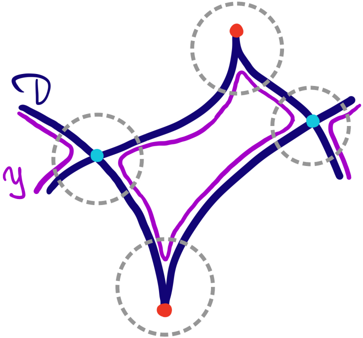
By the BKK theorem,

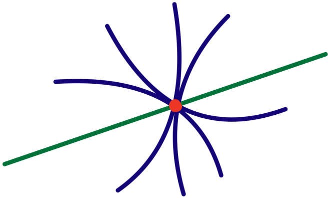
$\chi(Y) = -\operatorname{Area}(\mathcal{N}(\mathcal{D}))$

Thus, we have

$\chi(\mathcal{A}_1) - |\mathcal{A}_2| = -\operatorname{Area}(\underbrace{\mathcal{N}(\mathcal{D})}_{\text{known}})$

3 equations

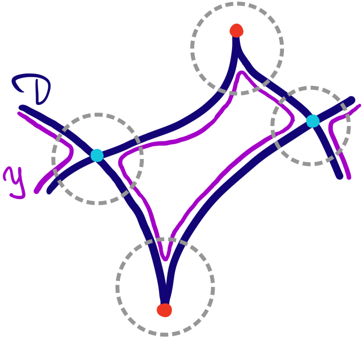


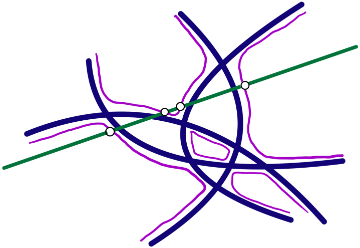


Thus, we have

$$\chi(\mathcal{A}_1) - |\mathcal{A}_2| = -\operatorname{Area}(\underbrace{\mathcal{N}(\mathcal{D})}_{\text{known}}) + \boxed{?!}$$

3 equations





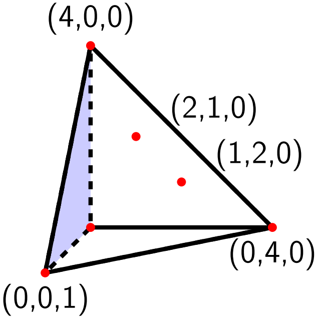
Proposition

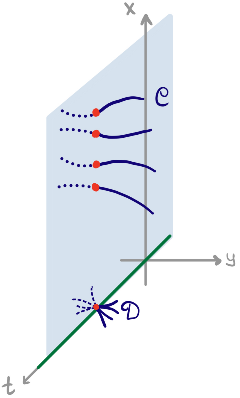
$$\chi(\mathcal{A}_1) - |\mathcal{A}_2| =$$

$$-\operatorname{Area}(\mathcal{N}(\mathcal{D})) - \sum_{s \in \mathrm{FPS}} \chi((\mathbb{C} \setminus 0)^2 \cap \textit{Milnor fiber of } s)$$

Singularities at infinity: example

$\mathcal{C} = \{f(x,y,t) = g(x,y,t) = 0\} \subset (\mathbb{C} \setminus 0)^3$ and $\mathcal{D} = \pi(\mathcal{C})$
f, g generic with support
$\tilde{A} = \{(0,0,0), (4,0,0), (2,1,0), (1,2,0), (0,4,0), (0,0,1)\}$.





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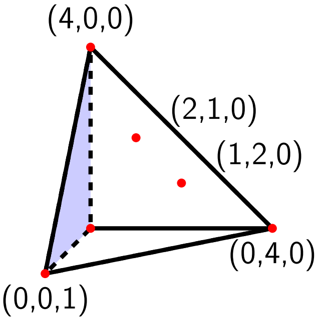

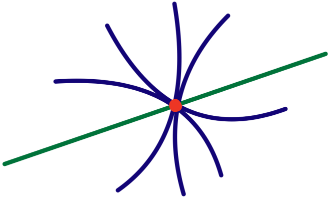

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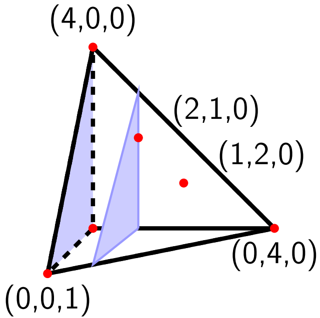

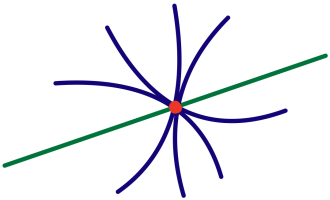

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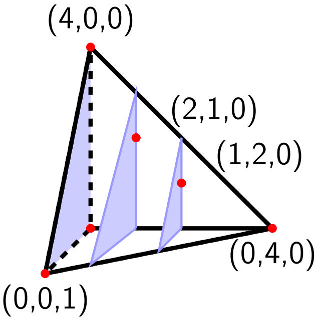

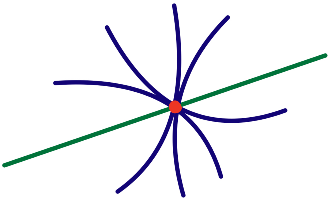

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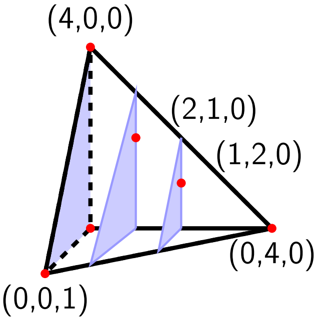
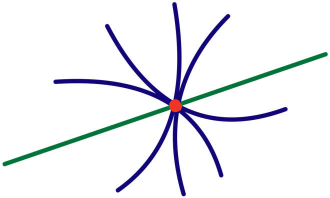


$$\mathrm{i} = (4, 2, 1, \ldots)$$



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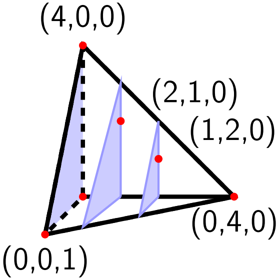

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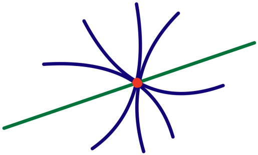
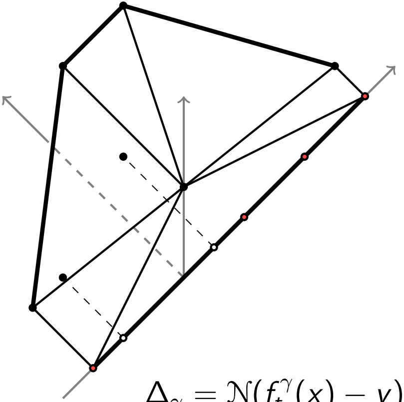


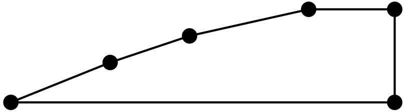
$4 - 4(3 + 2) = -16$

$$\chi(\text{Milnor fiber of } s) = \mathrm{i}_1 - \sum_{n=1}^{\infty} \mathrm{i}_1(\mathrm{i}_n - 1)$$

How it works in our case:

$\gamma\colon A \to \mathbb{Z}_{\geqslant 0}$ – a covector; $\pi\colon (x, y, t) \mapsto (y, t)$.





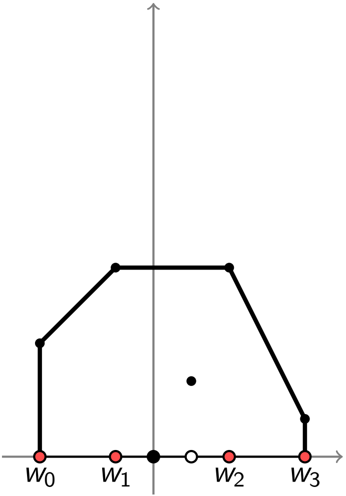
$$P = \int_\pi \Delta_\gamma$$

$$\Delta_\gamma = \mathcal{N}(f_t^\gamma(x) - y)$$

$$\tilde{A}_\gamma = \mathrm{supp}(f_t^\gamma(x) - y)$$

How it works in our case:

(Take $A = \{-3, -1, 1, 2, 4\}$ and $\gamma = (3, 5, 2, 5, 1)$)



$$W = \{w_0, w_1, w_2, w_3\} =$$
$$= \{-3, -1, 2, 4\};$$

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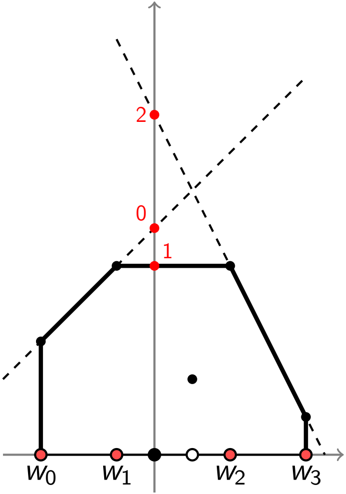

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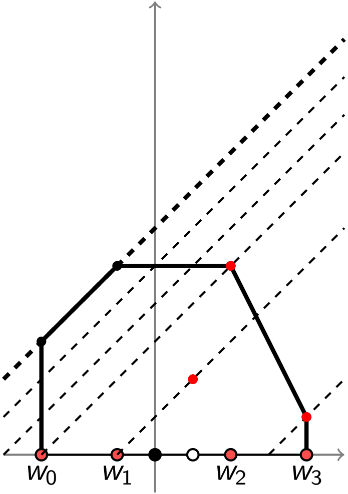


$$W = \{w_0, w_1, w_2, w_3\} = \\ = \{-3, -1, 2, 4\};$$

$$Z = (1, 0, 2);$$

$$M^0 = (2, 1, 4);$$

$$M^1 = (-3, 1, 4);$$

$$M^2 = (1, -1, -3).$$

3 equations

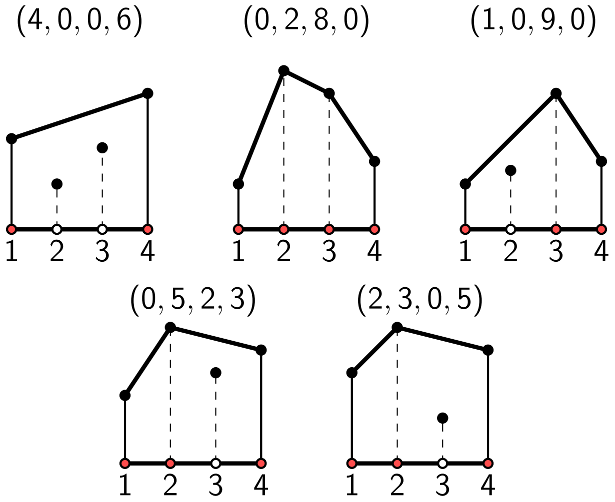
The sought number $|2\mathcal{A}_1|$ can be extracted from the following 3 equations:

$$|\mathcal{A}_2| = \mathrm{Area}(N_\gamma) - \gamma(a_0) - \gamma(a_{|A|-1})$$

$$\chi(\mathcal{A}_1) + 2|2\mathcal{A}_1| + 2|\mathcal{A}_2| = -\mathrm{Area}(N_\gamma)$$

$$\chi(\mathcal{A}_1) - |\mathcal{A}_2| = - \mathrm{Area}(\underbrace{\mathcal{N}(\mathcal{D})}_{\int_\pi \Delta}) - \sum_{s \in \mathrm{FPS}} \underbrace{\chi((\mathbb{C} \setminus 0)^2 \cap \text{Milnor fiber of } s)}_{\text{tricky, but we know how to compute it}}$$

Example revisited: $A = \{1, 2, 3, 4\}$



Thank you!!!

arXiv:2104.05123 [math.AG]