K-stability of singular dP surfaces

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① Introducing K-stability ; Why we care?

② Idea: Objects of Interest

③ K-stability — Obstructions
— Invariants : α, β, δ

④ Examples : 1. Index 2
2 eg of Index 3.

⑤ Future.

K-stability:

- Intro by G. Tian.
- Motivation : To understand the existence of KE metrics.

Fano Varieties:

Yau–Tian–Donaldson Conjecture:

Fano Variety X has KE metric $\Longleftrightarrow$ X is K-polystable.

K-stability introduced ' generalised Futaki invariant'.

Chen, Donaldson, Sun (Tian) : Proved YTD Conj for smooth Fano varieties

Li, Wang, Xu, Berman, Spotti, Sun, Yao : Proving YTD for KLT Fano.

② Objects of Today: Quasismooth / well-formed / hyp in WPS.

$$S_d \subset \mathbb{P}(\overset{x}{a_0}, \overset{y}{a_1}, \overset{z}{a_2}, \overset{t}{a_3})$$

d: deg of S

$a_0 \leq a_1 \leq a_2 \leq a_3$.

S: $f(x,y,z,t) = 0$

f is of deg d.

Quasi-smooth:

- singular only at the origin in $\mathbb{C}^4$.

$$p : \mathbb{A}^{n+1} \setminus \{0\} \xrightarrow[\text{can. projective}]{} \mathbb{P}$$

$$C^*_S = p^{-1}(S) \qquad S$$

affine cone.

$\mathbb{C}^*$ action on C^*_S.

Singularities on $C^*_S / \mathbb{C}^* = S$ are going to be given by $\mathbb{C}^*$.

$\Rightarrow$ $\boxed{\text{S can have at most cyclic quotient sing}}$

[Kollar]

$$-K_{S_d} \sim \mathcal{O}(a_1 + a_2 + a_3 + a_4 - d)$$

deg of S.

$[\, I = a_1 + a_2 + a_3 + a_4 - d > 0$

index of hypersurface.

$I > 0$.

$\Rightarrow -K_{S_d}$ ample.

Fano surface: dP surface

Well-formed : $S_d \subset \mathbb{P}$

$\mathbb{P}(a_0, a_1, a_2, a_3)$

is well-formed. if

① $\mathbb{P}$ is well formed.

$\gcd(a_1, a_2, a_3) = \gcd(a_0, a_2, a_3) \ldots = 1.$

② any codimension 2 singular strata of $\mathbb{P} \subset S_d$.

any singular curve of $\mathbb{P} \subset S_d$.

Classification Problem: $S_d \subset \mathbb{P}(a_0, a_1, a_2, a_3)$

$$I = a_0 + a_1 + a_2 + a_3 - d > 0.$$

Describe all possible such S_d that can exist.

$I = 1$. : Johnson Kollar [2001].

$I \geq 2$: Erik (Computer Code).

③ Obstructions? $X \subset \mathbb{P}(a_0, a_1, \ldots, a_n)$

$a_0 \leq a_1 \ldots \leq a_n$.

I. (Gauntlett, Martelli, Sparks, Yau) $I = \sum a_i - d$

X Fano Variety. admits no KE metric if either

① $I > n a_0$ //

or

② $dI^n > n^n \prod_{i=0}^{n} a_i$ //

Improved by Cheltsov, Shramov:

Let $a_0 \le a_1 \dots \le a_n$, $d \in \mathbb{R}_{>0}$

If $d\left(\hat{\sum_{i=0}^{n}} a_i - d\right)^n > n^n \prod_{i=0}^{n} a_i$

$$\Rightarrow \underbrace{\hat{\sum_{i=0}^{n}} a_i - d}_{I} > n a_0.$$

In our scenario $n = 3$.

Obstruction: S_d has no KE metric if

$I > 3a_0$.

or

$dI^3 > 27 \prod_{i=0}^{3} a_i$

Tian's criterion to show exis of KE metric

X Fano Variety

X admits a KE metric if

$$\alpha(X) = \operatorname{lct}(X) > \frac{\dim X}{\dim X + 1}.$$

$\operatorname{lct}(X) > \frac{2}{3}$.

$\operatorname{lct}(X) = \sup\{\lambda \in \mathbb{Q} \mid (X, \lambda D)$ is log canonical for any eff. $\underbrace{D \equiv -K_X}_{\mathbb{Q}\text{-div.}}\}$

$S_d \subset \mathbb{P}(\overset{x}{a_0}, \overset{y}{a_1}, \overset{z}{a_2}, \overset{t}{a_3})$

$I = \sum a_i - d.$

$$-K_X = \underbrace{\frac{I}{a_0}}\, \underset{\substack{\uparrow \\ x=0}}{H_x} \qquad \underset{\substack{\uparrow \\ a_0}}{I}\,H.$$

$$\mathrm{lct}(X, \lambda D) = \mathrm{lct}\left(X, \frac{I\lambda}{a_0} H_x\right)$$

By def^n (X, D) log canonical,

$$\Rightarrow \frac{\lambda I}{a_0} \leq 1$$

$$\lambda \leq \frac{a_0}{I}$$

$$\Rightarrow \mathrm{lct}(X, D) \leq \frac{a_0}{I}.$$

If $I \geq \frac{3a_0}{2}$

$$\Downarrow$$

$$\mathrm{lct}(X, D) \leq \frac{a_0}{I} < \frac{2}{3}$$

Cannot use Tian's criterion.

We want to look at cases $I < \frac{3a_0}{2}$, to use Tian's

(4) Invariants: β, δ.

Recall our setup!

S: dP surface w/ KLT sing.

$f : \tilde{S} \longrightarrow S$ bir. morphism.

E: prime div in $\tilde{S}$.

- $A_S(E) = 1 + \mathrm{ord}_E\left(K_{\tilde{S}} - f^*(K_S)\right)$ log discrepancy
- $S_{-K_S}(E) = \frac{1}{-K_S^2} \int_0^\tau \mathrm{vol}\left(f^*(-K_S) - uE\right) du.$

- $\tau(E) = \sup\{u \in \mathbb{Q} \,/\, f^*(-K_S) - uE \text{ is big}\}$.

  Pseudoeff threshold

D: effective $\mathbb{Q}$-div.

- **Volume**: ① $\mathrm{vol}(D) > 0 \iff D$ is big

  ② If D is nef $\Rightarrow \mathrm{vol}(D) = D^n$

  $n = \dim X$.

  ③ Zariski Decomposition:

  D: pseudoeff. div on surface S.

$$D = \underset{\text{pos}}{P} + \underset{\text{neg}}{N}.$$

(i) P is nef.

(ii) $N = \sum a_i N_i$ effective.

$\|N_i \cdot N_j\|$ neg. def.

(iii) $\forall i \; N_i \cdot P = 0$

$\dim S = 2$.

$$\mathrm{vol}(D) = \mathrm{vol}(P) = P^2$$

E: prime divisor over S.

β-invariant: $\beta(E) = \underbrace{A_S(E)} - \underbrace{S_{-K_S}(E)}$.

Thm: (Fujita, Li, Blum, Xu)

① X is K-st $\iff \beta(E) > 0$ $\forall$ prime div E over S.

② X is Kss $\iff \beta(E) \geq 0 \; \forall$ "

δ-invariant: $\delta(S) = \inf\limits_{E/S} \dfrac{A_S(E)}{S_{-K_S}(E)}$.

Thm: (Blum, Jonsson, Fujita, Li, Liu, Xu, Zhuang)

$\delta(S) \geq 1 \iff S$ is K-ss.

$\delta(S) > 1 \iff S$ is K-st.

K-stability $\Longrightarrow$ K-ps $\Longrightarrow$ K-ss.

Local Analogue of δ: Pt $p \in S$.

$$\delta_p(S) = \inf_{\substack{E/S \\ p \in C_S(E)}} \frac{A_S(E)}{S_{-K_S}(E)}$$

$$\delta(S) = \inf_{p \in S} \delta_p(S).$$

Abban–Zhuang Theory: (setup)

S: dP surface w/ atmost KLT.

Y: irred curve on S.

Thm: [Abban, Zhuang, ACCFKGSSV]

$$\delta_p(S) \geq \min\left\{ \frac{1}{S_S(Y)}, \frac{A_Y(p)}{S(W_{\cdot,\cdot}^Y; p)} \right\}$$

above

$$S(W_{\cdot,\cdot}^Y; p) = \frac{2}{-K_S^2} \int_0^\tau h(u)\,du.$$

$$h(u) = (P(u)\cdot Y)\, \mathrm{ord}_p(N(u)|_Y) + \int_0^\infty \mathrm{vol}(P(u)|_Y - vp)\,dv$$

⑤ What do we know?

$\underline{I=1}$: (J, K, A, C, P, S) $I=1$

except $S_{15} \subset \mathbb{P}(1,3,5,7)$ w/ $yzt \notin f(x,y,z,t)=0$

$\Rightarrow \alpha > \frac{2}{3}$ $\therefore \exists$ KE metric.

$\to \alpha < \frac{2}{3}$.

(Cheltsov, Park, Won): For exception:

$$\delta(S_{15}) \geq \frac{6}{5} > 1.$$

$\Rightarrow$ K-stable $\Rightarrow$ K-ps $\therefore \exists$ KE metric.

$\underline{I=1}$ $\Rightarrow \exists$ KE metric.

$\underline{I=2}$: [B, G, N, C, P, S]: Many of infinite series, sporadic. $\Rightarrow \exists$ KE metric.

(In-Kyun Kim, Won): $I=2$. 6 diff. hyp. for which $\nexists$ KE metric

Index 2: Our work:

Theorem 1.0.6 (**Main Theorem**). *Let S_d be a quasi-smooth, well-formed hypersurface with $I = 2$. The following table gives our results on the existence of Kähler-Einstein metrics on S_d in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree d.*

No.	(a_0, a_1, a_2, a_3)	degree	KE
1	$(1, 1, n, n)$, $n \geqslant 2$	$2n$	yes
2	$(1, 2, n+2, n+3)$, $n \geqslant 0$	$2(n+3)$	yes $\to$ δ: AZ theory.
3	$(1, 3, 4, 6)$	12	yes
4	$(1, 4, 5, 7)$	15	yes
5	$(1, 4, 5, 8)$	16	yes
6	$(1, 4, 6, 9)$	18	yes
7	$(1, 5, 7, 11)$	22	yes
8	$(1, 6, 10, 15)$	30	yes
9	$(1, 7, 12, 18)$	36	yes
10	$(1, 8, 13, 20)$	40	yes

I = 3: [BGN] : Sporadic cases.

[CPS] : 2010.

I = 3 Our work:

No.	weights	degree	KE
1	$(1, 2, 2n+3, 2m+3)$	$2(n+m)+6$	No
$1^\dagger$	$(1, 2, 2n+3, 2n+3)$	$4n+6$	Yes $\to$ β
2	$(1, 1, 2, 2n+3)$	$2n+4$	No
3	$(1, 5, 10n+5, 10n+7)$	$20n+15$	No
4	$(1, 5, 10n+7, 10n+9)$	$20n+19$	No
5	$(1, 7, 9, 13)$	27	No
6	$(1, 7, 9, 14)$	28	No
7	$(1, 9, 13, 20)$	40	No
8	$(1, 13, 22, 33)$	66	No
9	$(1, 14, 23, 35)$	70	No
10	$(1, 15, 25, 37)$	75	No

| lct.

I = 2:

$$\underbrace{S_{2(n+3)}}\subset \mathbb{P}(\underset{x}{1}, \underset{y}{2}, \underset{z}{n+2}, \underset{t}{n+3}) \qquad n \geq 0.$$

$$f(x,y,z,t) = t^2 + z^2 y + f_{2n+6}(x,y) = 0.$$

Singular pts: n even:

$p_z : [0:0:1:0]$ $\frac{1}{n+2}(1,1)$

$Q_1, Q_2 : [0:1:\alpha:0]$ $\frac{1}{2}(1,1)$

n odd $[0:0:0:1]$ Pt ; Q_1', Q_2' $[0:1:0:\alpha]$

$\frac{1}{n+2}(1,1)$ $\underbrace{}\frac{1}{2}(1,1)$

Obj: Compute $\delta(S)$.   (Local Analogue)

$p \in S$; Y: curve on $S \ni p$.

$$\underset{pt}{p} \in \underset{curve}{Y} \subset \underset{surface}{S}$$

Case 1: $p \in C_x$: $C_x = \underset{x=0}{H_x} \cap S_{2(n+3)}$

p: sing pts p: smt. pt $\tilde{S} \longrightarrow S$

$$\underbrace{-K_S - uC_x}$$
$$= 2C_x - uC_x.$$
$$= (2-u)C_x.$$

$K_S^2 = ? = \frac{4}{n+2}$

$K_S = 2C_x$

$C_x^2 = \frac{1}{n+2}$

$u \le 2$ $\tau(u) = 2$.

$A_S(C_x) = 1 + \mathrm{ord}_p(\underline{K_{\tilde S} - f^*K_S})$

$= 1$

$S_S(C_x) = \frac{1}{K_S^2}\int_0^2 \mathrm{vol}(-K_S - uC_x)\,du$

$= 2/3$

$$\frac{A_S(C_x)}{S_S(C_x)} = \frac{3}{2}$$

$$\delta_p \geq \min\left\{\frac{1}{S_S(C_x)}, \frac{A_p(E)}{\sim\sim}\right\}$$

p = smt pt

$A = 1.$

$S(W^{Y}_{\cdot,\cdot}; p) = ?.$

$p : \mathbb{C}^2/\mathbb{Z}_m.$

(Shokurov 3-fold log Flips).

coeff Δ

p = sing pt

$p = p_2$

$[0:0:1:0]$

$\frac{1}{n+2}(1,1)$

$$A_p(C_x) = ?$$

$$= 1 - \left(\frac{m-1}{m}\right)$$

$$= 1 - \left(\frac{n+2-1}{n+2}\right)$$

$$= \frac{1}{n+2}.$$

$$S(W^{Y}_{\cdot,\cdot}; p) = \frac{2}{-K_S^2}\int_0^{\tau} h(u)\,du.$$

$$h(u) = (P(u).C_x)\underbrace{\operatorname{ord}_p\left(N(u)\big|_{C_x}\right)}_{} = 0$$

$$+ \int_0^{\infty} \operatorname{vol}\left(P(u)\big|_{C_x} - v_p\right) dv.$$

$D = P + N$

D 0

$$-K_S - uC_x = (2-u)C_x. \leftarrow \text{Nef.}$$

$0 \leq u \leq 2$

effective

Nefness.

$$(2-u)C_x \cdot C_x = \frac{2-u}{n+2} > 0$$

$$= \int_0^2 \int_0^\infty \mathrm{vol}\left(P(u)\big|_{C_x} - v_p\right) dv \, du .$$

$$(2-u) C_x^2 = \frac{2-u}{n+2}$$

$$= \int_0^2 \int_0^{\frac{2-u}{n+2}} \frac{2-u}{n+2} - v \, dv \, du$$

$$= \frac{2}{3(n+2)} .$$

$$\delta_p \geq \min \left\{ \frac{3}{2} , \frac{\frac{1}{n+2}}{\frac{2}{3(n+2)}} \right\}$$

$\underline{p \in C_x}$.

$$\geq 3/2$$

$\underline{p \notin C_x}$: $\boxed{[1 : y : z : t]}$ $\mathbb{P}(1, 2, n+2, n+3)$

$\underline{\text{Eq}^n \text{ of } S}$: $y = \mu x^2 . \ni P .$

Step 1:

$\mathrm{lct}(S)$

Step 2: (Blum, Jonsson) X : Fano dim n

$$\frac{\dim X + 1}{\dim X} \alpha(X) \leq \delta(X) \leq (\dim X + 1) \alpha(X)$$

3.