# Munich talk - 17/01/2024

Expansions for Hilbert schemes of points on semistable degenerations.

**Aim**: Construct "good" degenerations of Hilbert schemes of points over semistable degenerations of surfaces.

Motivating example: Hilbert schemes of points on maximally degenerate K3 surfaces.

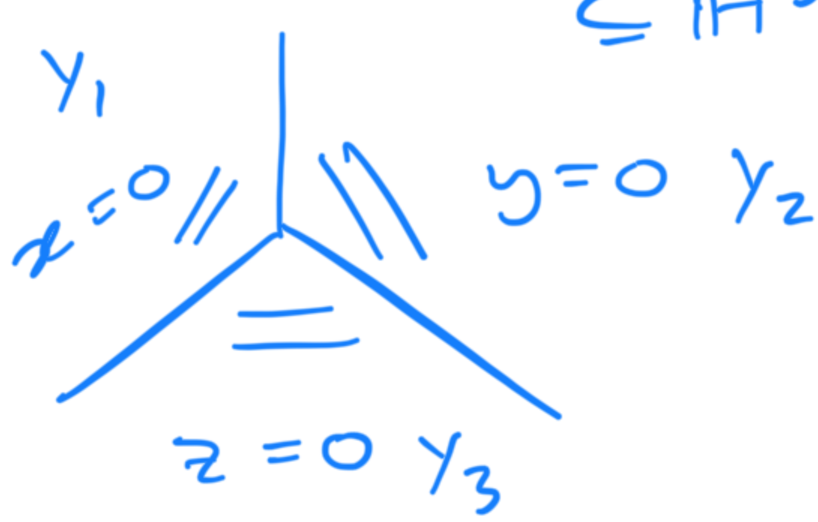
Basic setup:

Let $X \to \overset{\cong \mathbb{A}^1}{C}$ be flat projective family of surfaces & consider étale local model given by $\operatorname{Spec} k[x,y,z,t]/(xyz-t)$:

alg. clsd, char. 0

- general fibres smooth,
- special fibre $X_0$ over $0 \in C$ has SNC singularity.

$X_0 \subseteq \mathbb{A}^3$




Now, let $X^o := X \setminus X_0$

$$C^\circ := C \setminus \{0\}$$

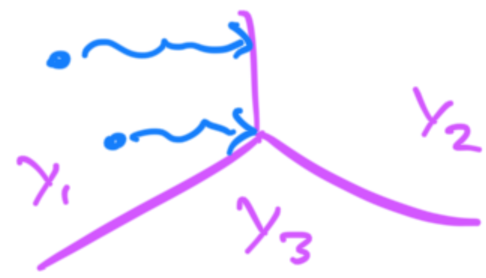
Rephrasing aim: Construct a "good" compactification of relative Hilbert scheme of points $\mathrm{Hilb}^m(X^\circ / C^\circ)$.

Meaning of "good":

- Hyperkähler perspective:
  Let $X \to C$ be a type III (max degen.) good degeneration of K3 surfaces.
  $\rightsquigarrow$ want to construct a type III degen of Hilb. schemes of points on K3 surfaces which is minimal wrt to MMP.
  $\downarrow$
  here dlt minimal as scheme or semistable minimal as stack.

- Moduli perspective:
  $\rightsquigarrow$ want to construct a flat degen (as proper D-M stack) where limit of each family of length $m$ 0-dim subschemes has smooth support in compactification.

e.g. if we have point tending towards intersection locus of $X_0$: we want to modify $X_0$ so that they land in sm locus of a component





$\Rightarrow$ Consequence of making such a constr$^n$:
All data of degeneration is contained in limit.

## Example to illustrate situation

Take a family of length 5 0-dim subschemes in Spec $k[x,y,z,t]/(xyz-t)$,

$$Z = \overset{Z_1}{\{x-y=0,\ z=1\}} \cup \overset{Z_2}{\{x-y^2=0,\ z=1\}}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

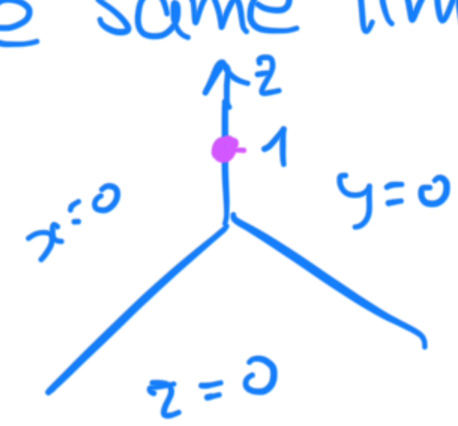
$$x^2 = t = u^6 \qquad\qquad y^3 = t = u^6$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\{(\underset{x}{u^3}, \underset{y}{u^3}, \underset{z}{1})\} \cup \{(\underset{x}{u^4}, \underset{y}{u^2}, \underset{z}{1})\}$$

$\downarrow$

Both points of support have same limit in compactification:




But they are clearly

different ("approach singular locus at different speeds"). We want the compactification to reflect this.

↳ Their degrees of vanishing in $x$ & $y$ are different. This is what we need to record.

$\Downarrow$

Need to make modifications of $X_0$ where both these points land in different components. How?

(limit points of the support corresponding to $z_1, kz_2$)

Tropicalisation + expanded degen's.

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Previous work in this area:

- Li (2001): Introduces notion of expanded degen's.
- Li-Wu (2011): construct good degenerations of Quot schemes over $X \to C$ where sing$(X_0)$ is smooth.
- Gulbrandsen-Halle-Hulek (2017): Describe GIT analogue of Li-Wu construction for Hilbert schemes of points.
- Maulik-Ranganathan (2020): use log & tropical geometry to construct good degens of Hilbert schemes for $X \to C$ where $X_0$ has any type of SNC singularity.
- Kennedy-Hunt (2023): Extends MR to Quot schemes.

$\Downarrow$

What remains to be done?
Have MR not solved the problem?

Yes, BUT : • their results yield an infinite family of birational solutions;

• we don't know what these solutions look like : they give no explicit model & it is not clear what an explicit model would look like.

We will show how to construct geometrically meaningful models. We start by explaining insights & limitations of MR:

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Q1) How to construct modifications so that limits of subschemes have smooth support in modification?

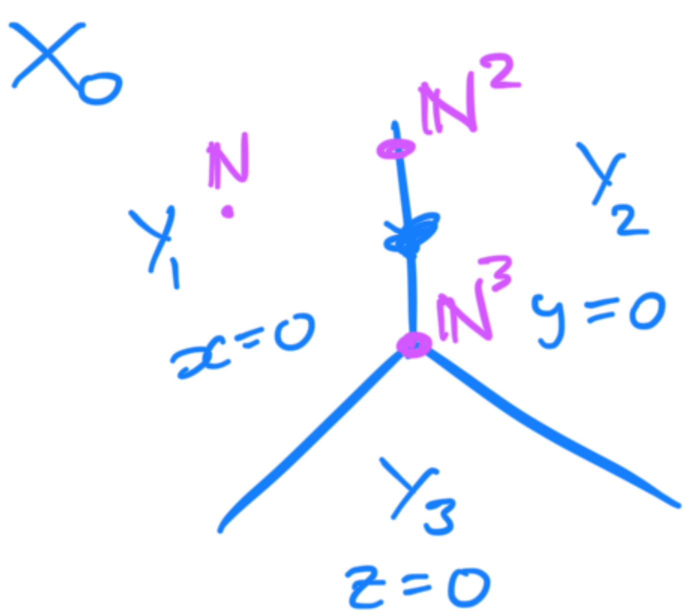
Q2) How to fit all these modifications into 1 big family?

<u>Tropicalisation</u>

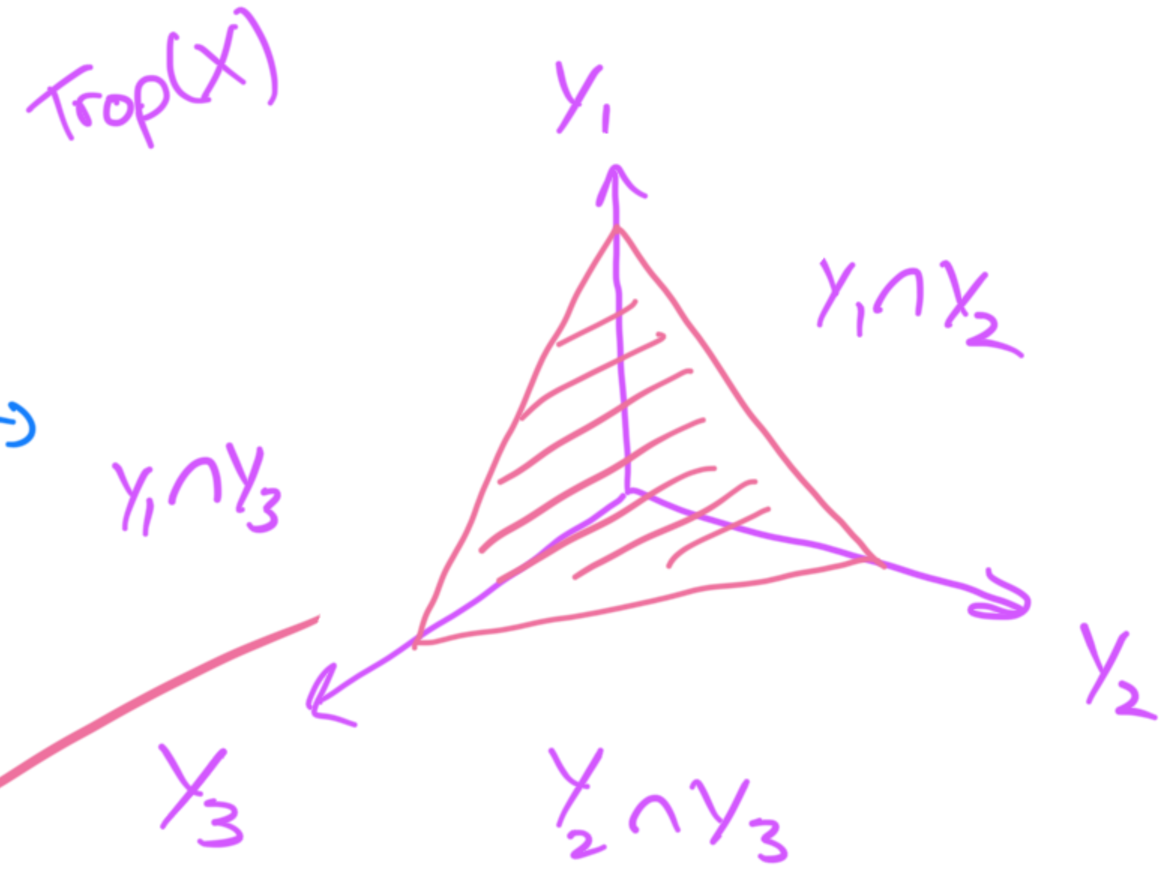
In example, we saw that we want to keep track of deg of vanishing in $x, y, z$ for each subscheme in $\mathrm{Hilb}^m(X^\circ/C^\circ)$.

$\rightsquigarrow$ Build tropicalisation of $X$ w.r.t. divisor $X_0$

functions $x, y, z$ vanish here. Tropicalisation will record the deg of vanishing of these functions.



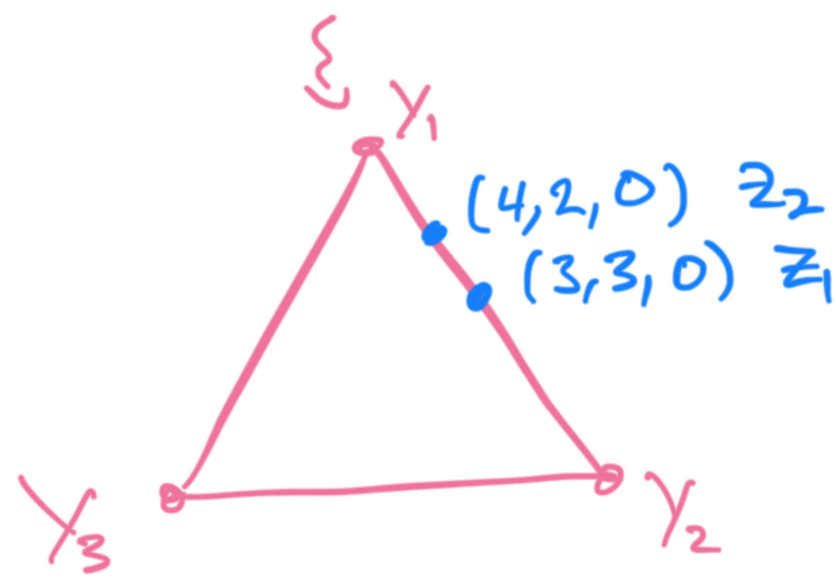

$\rightsquigarrow$




Trop($X_0$)
↳ dual complex of $X_0$

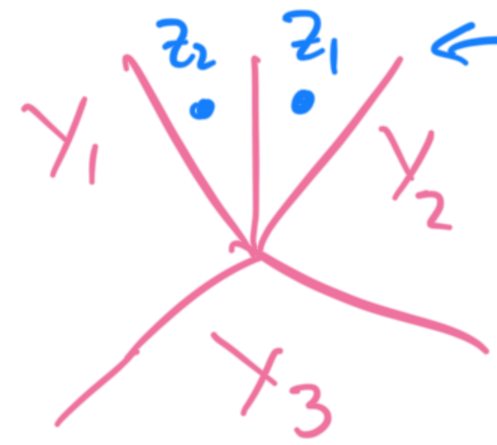
Back to example:

$$\begin{array}{cccc} & \scriptstyle x\quad y\quad z & & \scriptstyle x\quad y\quad z \\ Z = & \{(u^3,u^3,1)\} & \cup & \{(u^4,u^2,1)\} \\ & Z_1 \qquad \downarrow & & \downarrow \qquad Z_2 \\ & (3,3,0) & & (4,2,0) \end{array}$$


↳ This tells us that if we want to represent limit of $Z$ with smooth support, we need to add 2 components to $X_0$ in $Y_1 \cap Y_2$

locus




limit of $z_1$ & $z_2$ land in interior of component corresponding to their image in tropicalisation.

$\Rightarrow$ This tells us exactly how to modify $X_0$ to represent limit with smooth support!

$\downarrow$

Now I just need to exclude all "bad" subschemes, i.e. subschemes which are not smoothly supported:

Li-Wu stability: A subscheme $Z$ is stable if it is smoothly supported & each added component contains a point of the support of $Z$. (We actually add a $\mathbb{G}_m$-action on each new component, so leaving one empty gives infinite stabilisers & we don't want that.)

Problems:

- These modifications are not blow-ups! (If we try to fit all these into a big family it will not be flat).

$\downarrow$

We need to adjust the modifications to make them be blow-ups

↳ • breaks separatedness & we need to impose additionally Donaldson–Thomas stability to fix this.

↳ involves a lot of choices!

∘ For each $z$ a modification was individually constructed but we don't know how they fit together.

↳ This is HARD in MR

Again involves a lot of choices & it is not easy to see what the resulting object looks like.

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## My work

- Explicitly construct a family of modifications:

$$\begin{tikzcd} X \times_{\mathbb{A}^1} \mathbb{A}^{n+1} \arrow[r] \arrow[d] & \mathbb{A}^{n+1} \ni (t_1, \dots, t_{n+1}) \arrow[d] \\ X \arrow[r] & C \cong \mathbb{A}^1 \ni t = t_1 \cdots t_{n+1} \end{tikzcd}$$

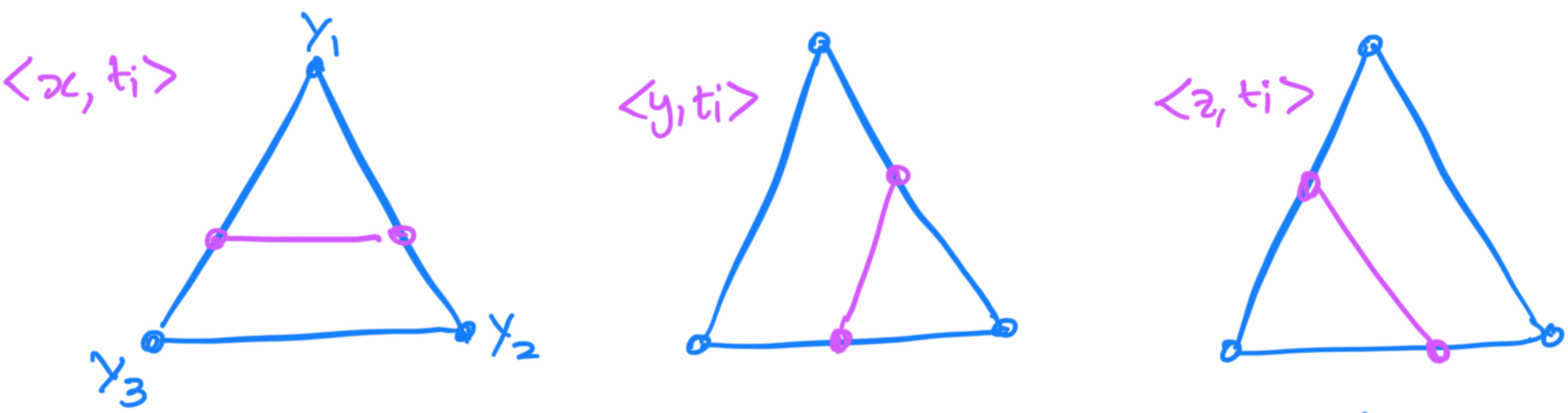
↳ This gives us large family with many copies of $X_0$.

$\operatorname{Spec} k[x,y,z,t]/(xyz-t) \to \operatorname{Spec} k[t]$

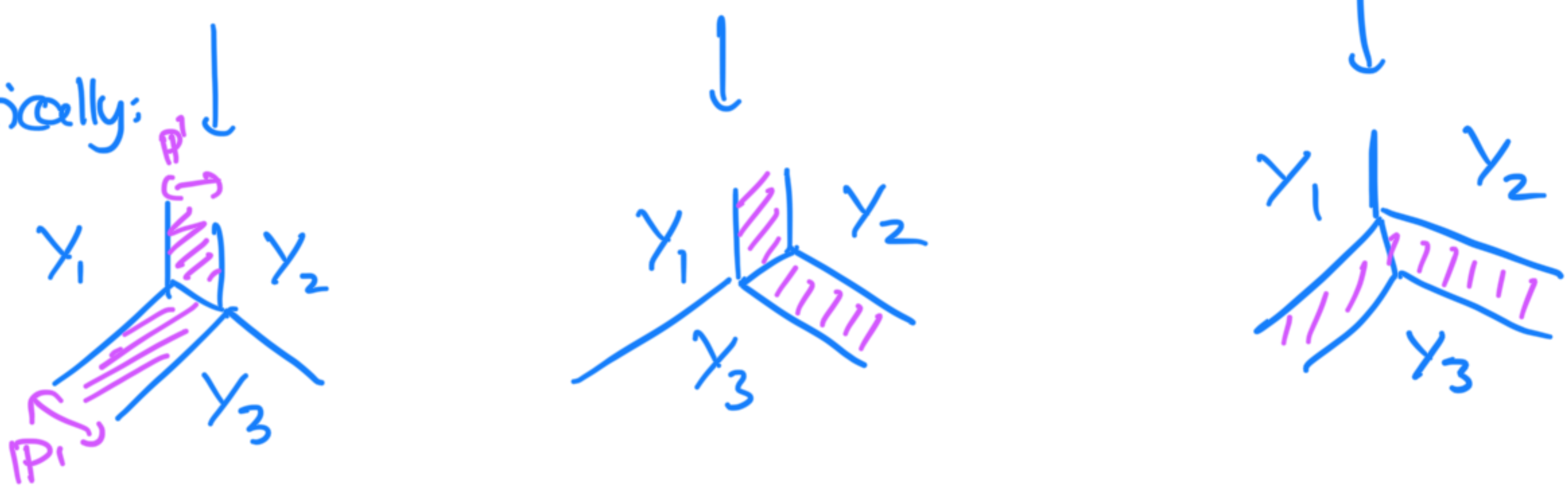
- Make sequence of blow-ups on $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ :

• Take sequence of blow-ups on X in

we only allow blow-ups of type:




Geometrically:



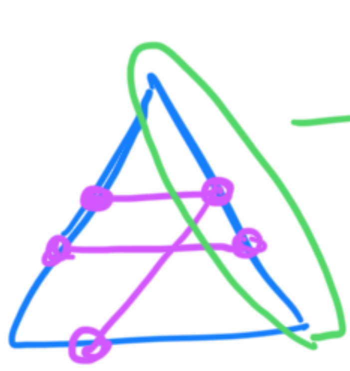

For any configuration of vertices in $\Delta$ given by $\mathbb{Z}$, I need to be able to add edges so that each vertex lands on intersection of at least 2 edges.

↓

I only need 2 of the above types to make this work, e.g.  & .

⟶ Break symmetry

⟶ Our blow-ups commute so they fit nicely into big family.



→ All data of blow-ups is contained in this edge: for each integral point, look if it has ⟶

or $\vee$ / edge attached.

(*) { Each integral point must have no edge or both — & / attached.

Then call the family of modifications obtained $X[n]$.

Add a group action to get us back to 1-parameter family & make into stack $\mathfrak{X}$.

Li-Wu or modified GIT
↓
<u>Thm</u> The stack of <u>stable</u> length m 0-dim subschemes in $\mathfrak{X}$ is DM & proper over $C$.

→ No need for DT stab$^y$! This is unexpected consequence of (*) property.

If we don't impose $(*)$, we allow more mod$^n$s, larger family $\mathfrak{X}'$,

$\Rightarrow$ Li-Wu stable stack no longer separated

identify limits of some family

proper non-alg stack, parallel with Kennedy-Hunt

add extra stab$^y$ cond$^n$ to cut out proper substack

Recover some choices of MR stacks