A. The Problem

Prb1 [unlabeled PCA]

$\mathcal{X} \in \mathbb{R}^{m \times n}$, $\operatorname{rank}(\mathcal{X}) = r$

σ: permutation of matrix entries

$\tilde{\mathcal{X}} = \sigma(\mathcal{X})$, What can we say about the recovery of $\mathcal{X}$ from $\tilde{\mathcal{X}}$? Can there be efficient algorithms? $\blacksquare$

Rem 2 At best, $\mathcal{X}$ can be recovered from $\tilde{\mathcal{X}}$ up to a row and column permutation $\Pi_1 \mathcal{X} \Pi_2$ $\blacksquare$

Rem 3 unlabeled PCA

$$\begin{array}{cc} \text{unlabeled} & \text{PCA} \\ \nearrow & \uparrow \\ \text{unlabeled sensing} & \text{PCA} \end{array}$$

$\blacksquare$

B. Principal Component Analysis

$\tilde{X} = [\tilde{x}_1 \cdots \tilde{x}_s] \in \mathbb{R}^{m \times s}$

goal: find $\mathcal{V} \subset \mathbb{R}^m$ of dimension r
s.t. the $\tilde{x}_j$'s are close to $\mathcal{V}$

no noise $\quad r = \operatorname{rank}(\tilde{X})$
$\mathcal{V}$: column space of $\tilde{X}$
SVD

robust PCA

X: rank-r ground truth matrix, $\tilde{X}$: corruption of X

i) "sparse noise" $\tilde{X} = X + E$
sparse

$$\boxed{\min_L \|L\|_* + \lambda \|\tilde{X} - L\|_1}$$

ideally $L = X$

ii) "outliers"
permutation

$\tilde{X} = [X \; O] \Pi$

$$\min \|L\|_* + \lambda \|Y\|_{2,1}$$
$$\tilde{X} = L + Y$$

works if r is small

ideally $L = [X \ \ 0]$

$Y = [0 \ \ 0]$

Algorithm A

$$\min_{B \in \mathbb{R}^{m \times c}} \|\tilde{X}^T B\|_{1,2} \quad \text{s.t.} \quad B^T B = I_c$$

works for any r, but non-convex

$c = \operatorname{codim} \mathcal{C}(X)$

$= m - r$

iii) "missing entries" $\Leftrightarrow$ low-rank matrix completion

$$\tilde{X} = X \odot \Omega$$

0,1 observation pattern

$$\min_{L} \|L\|_* \quad \text{s.t.} \quad \tilde{X}_{ij} = L_{ij} \quad (i,j) \in \Omega$$

connections with algebraic geometry, commutative algebra combinatorics

Dfn 4 κ: infinite field

$\mathbb{M}(r, m\times n) = \{X \in \kappa^{m\times n} : \mathrm{rank}(X) \leq r\}$

$\mathbb{A}^{\Omega}$: $m\times n$ matrices with support in Ω

coordinate projection

$\pi_{\Omega} : \mathbb{M}(r, m\times n) \longrightarrow \mathbb{A}^{\Omega}$

Prb 5 [algebraic matroid of $\mathbb{M}(r, m\times n)$]

Which π_{Ω}'s have finite generic fiber? $\boxtimes$

iv) "permutations"

G. Unlabeled Sensing

UnniKrishnan et al. '15, '18

$Ax = b$

$m\times r$

$m > r$

full rank

unique solution }

permutation

$\pi : \mathbb{R}^m \longrightarrow \mathbb{R}^m$

coordinate projection

$\rho : \mathbb{R}^m \longrightarrow \mathbb{R}^s$

Prb 6 [unlabeled sensing]

Do the data A, $\rho \circ \pi(b)$ uniquely define ξ?

Thm 7 [UHV, '15] Yes, if A is generic and $s \geq 2r$

rank of ρ

dimensionality of solution ξ

Prb 8 [homomorphic sensing]

$\mathcal{T}$: finite set of endomorphisms

$\tau_i : \mathbb{K}^m \to \mathbb{K}^m$

$\mathcal{V}$: linear subspace of $\mathbb{K}^m$

$\dim \mathcal{V} = r$

$v_1, v_2 \in \mathcal{V}$ $\quad$ $\tau_1, \tau_2 \in \mathcal{T}$ $\quad$ HSP

Under what conditions

$\tau_1(v_1) = \tau_2(v_2) \Rightarrow v_1 = v_2$?

Thm 9 [Peng, T., '18–'20]

Y_{τ_1,τ_2}: variety of K^m defined by the 2-minors of $[\,T_1 z \quad T_2 z\,]$

$z = \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix}$

$U_{\tau_1,\tau_2} =$ locally closed

$Y_{\tau_1,\tau_2} \setminus \mathrm{Ker}(\tau_1) \cup \mathrm{Ker}(\tau_2) \cup \mathrm{Ker}(\tau_1 - \tau_2)$

$V \in \mathrm{Gr}(r,m)$ generic and $\underbrace{\dim V}_{r} \leq \operatorname{codim} U_{\tau_1,\tau_2} \Rightarrow$ HSP ✓ $\blacksquare$

Rem 10 Also extends to subspace arrangements $\blacksquare$

Rem 11

Algorithms $\to$
- dynamic programming
- branch & bound
- concave minimization
- ℓ_1-regression (Slawski & Ben-David)
- algebraic geometry

$\blacksquare$

$$\underset{m\times r}{A}x = b \quad , \quad y = \sigma(b)$$

$$A, y \mapsto \xi$$

$$K[z] = K[z_1, \dots, z_m]$$

$$p_\ell(z) = z_1^\ell + \dots + z_m^\ell \quad , \quad \ell = 1, \dots, r$$

$$p_\ell(A\xi) = p_\ell(y)$$

$\leftarrow$ r constraints that the solution $\xi \in K^r$ satisfies

$\Bbbk[w] = \Bbbk[w_1, \ldots, w_r]$

$\hat{p}_\ell(w) = p_\ell(Aw) - p_\ell(y)$

$\ell = 1, \ldots, r$

Thm 12 [Choi, Conca, T., '18]

A generic $\Rightarrow$ the variety $\mathbb{V}(\hat{p}_1, \ldots, \hat{p}_r) \subset \Bbbk^r$ consists of finitely many points.

Prf $p_1(z), \ldots, p_r(z)$: regular sequence of $\Bbbk[z_1, \ldots, z_m]$ + Gröbner bases theory for weighted term orders $\Rightarrow$

the $\hat{p}_i$'s are non-homogeneous

$\hat{p}_1(w), \ldots, \hat{p}_r(w)$: regular sequence of $\Bbbk[w_1, \ldots, w_r]$ $\boxtimes$

Alg 13 [Choi, Conca, Kneip, Peng, Shi, T.; '18]

Input: $\mathcal{V}$, y → some permutation of some $b \in \mathcal{V}$

r-dimensional linear subspace

Output: b

Algorithm B

Solve the polynomial system

$p_\ell(Aw) - p_\ell(y) = 0$, $\ell = 1, \dots, r$

at most $r!$ roots → best root $\hat{\xi}$

$$\underset{\Pi', x}{\operatorname{argmin}} \|\Pi' y - Ax\|_2$$

alternating minimization initialized with $\hat{\xi}$ ☐

Ex 14 $r = 4$, $m = 1000$, SNR = 40dB

0.4% estimation error in 15 msec ☐

Thm 15 [Liang, Lu, T., Zhi, '23]

A: generic $m\times n$, $b=A\xi$,

ξ: generic in K^r, $y=\sigma(b)$ $\Rightarrow$

The variety

$\mathbb{V}(\hat{p}_1(w),\dots,\hat{p}_{r+1}(w)) \subset K^r$

consists only of ξ (set-theoretically)

Prb 16 Efficient and robust algorithm for solving

$p_\ell(Aw)-p_\ell(y)=0$, $\ell=1,\dots,r+1$? $\boxtimes$

D. Unlabeled PCA

Thm 17 [T., '22]

X: generic in $\mathcal{M}(r, m\times n)$

σ: any permutation of matrix entries

Then $\operatorname{rank}(\sigma(X)) = r \iff$

$\sigma(X) = \Pi_1 X \Pi_2$ or

$\sigma(X) = X^T$, if $m=n$.

up to a row and column permutation, X is the only rank-r matrix that agrees with $\sigma(X)$

Prf [sketch]

$Z = (z_{ij})$: $m\times n$ matrix of variables

$k[Z]$: polynomial ring of dimension mn

$I_{r+1}(Z)$: ideal of $(r+1)$-minors of Z

Thm 18 [Narasimhan '86; Sturmfels '90; Caniglia, Guccione J.A/J.J '90; Ma '94; Sturmfels, Sullivant '06]

The $(r+1)$-minors of Z are a Gröbner basis for $I_{r+1}(Z)$ under <u>any</u> diagonal or anti-diagonal term order $\boxtimes$

$\Rightarrow$ the set of $X \in \mathbb{M}(r, m\times n)$ s.t. $\sigma(X) \in \mathbb{M}(r, m\times n)$ is a proper <u>subvariety</u> of $\mathbb{M}(r, m\times n)$, except when σ permutes <u>only</u> rows and columns $\boxtimes$

$$p_\ell(Z) = \sum_{\substack{i=1,\dots,m \\ j=1,\dots,n}} z_{ij}^{\ell} \quad , \ell=1,\dots,mn$$

J = ideal of $k[Z]$ generated by

$$\hat{p}_\ell(Z) = p_\ell(Z) - p_\ell(X), \ \ell=1,\dots,mn$$

<u>Rem 19</u> With X generic, it is easy to see by Thm 19 that the variety $\mathbb{V}(I_{r+1}(Z), J)$ of $k^{m\times n}$ consists only of X and its row+column permutations $\blacksquare$

<u>Rem 20</u> The variety $M(r, m\times n)$ is irreducible of dimension $r(m+n-r)$ $\blacksquare$

Thm 21 [T., '22]

$\mathcal{X}$: generic in $\mathbb{M}(r, m\times n)$

σ : any permutation of matrix entries

Then $r(m+n-r)+1$ generic linear combinations of $\hat{p}_1, \dots, \hat{p}_{mn}$ cut $\mathbb{M}(r, m\times n)$ set-theoretically at all points $\Pi_1 \mathcal{X} \Pi_2$ (and $\mathcal{X}^T$ if $m=n$)

Prf [sketch] Follows from:

Thm 22 [Hochster & Eagon, '71]

The ring $k[Z]/I_{r+1}(Z)$ is Cohen-Macaulay $\blacksquare$

Prp 23 $\mathcal{A}$: Noetherian ring that contains an infinite field $\mathbb{K}$, $I = (\alpha_1, \ldots, \alpha_s)$ ideal of $\mathcal{A}$ s.t. $\operatorname{grade}(I) > 0$. Then a linear combination $\alpha = c_1\alpha_1 + \cdots + c_s\alpha_s$ with the c_i's chosen generically in $\mathbb{K}$ is $\mathcal{A}$-regular $\boxtimes$ $\boxtimes$

E. A Special Case of UPCA

[Yao, Peng, T., '21]

"column-wise permutations with a dominant permutation"

$\mathcal{X} = [x_1 \cdots x_n] \in \mathbb{R}^{m \times n}$

$\pi(\mathcal{X}) = [\pi_1(x_1) \cdots \pi_n(x_n)]$

σ_j : permutes j-th column of X

Dfn 24 multiplicity $\mu(\sigma_j)$:
$\#\{j' : \sigma_{j'} = \sigma_j\}$ ⧄

Ass 25 $\exists\ j$ s.t. σ_j is dominant,
i.e. $\mu(\sigma_j) \gg \mu(\sigma_{j'})\ \forall\ \sigma_{j'} \neq \sigma_j$
⧄

Rem 26 $\mathcal{C}(X)$: column-space of X

$$\sigma(X) = [\tilde{X}'\ \ O]\,\Pi$$

outliers

the columns
lie in $\sigma_j(\mathcal{C}(X))$ ⧄

Alg 27 [UPCA]

estimate of $\sigma_j(c(x))$

$$\sigma(x) \to \boxed{\text{Alg.A}} \xrightarrow[\sigma(x)]{\nu} \boxed{\text{Alg.B}} \to \hat{x}$$

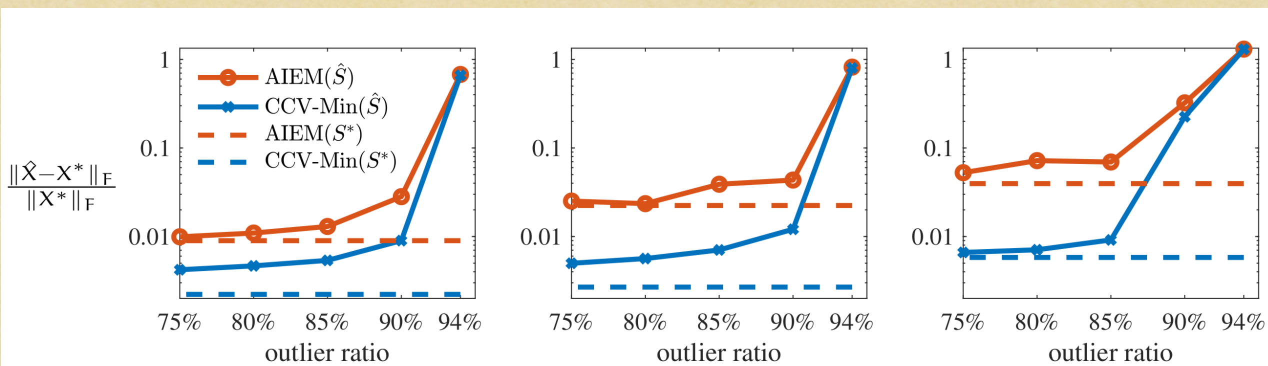
PCA with robustness to outliers

Unlabeled Sensing

estimate of x

$\blacksquare$

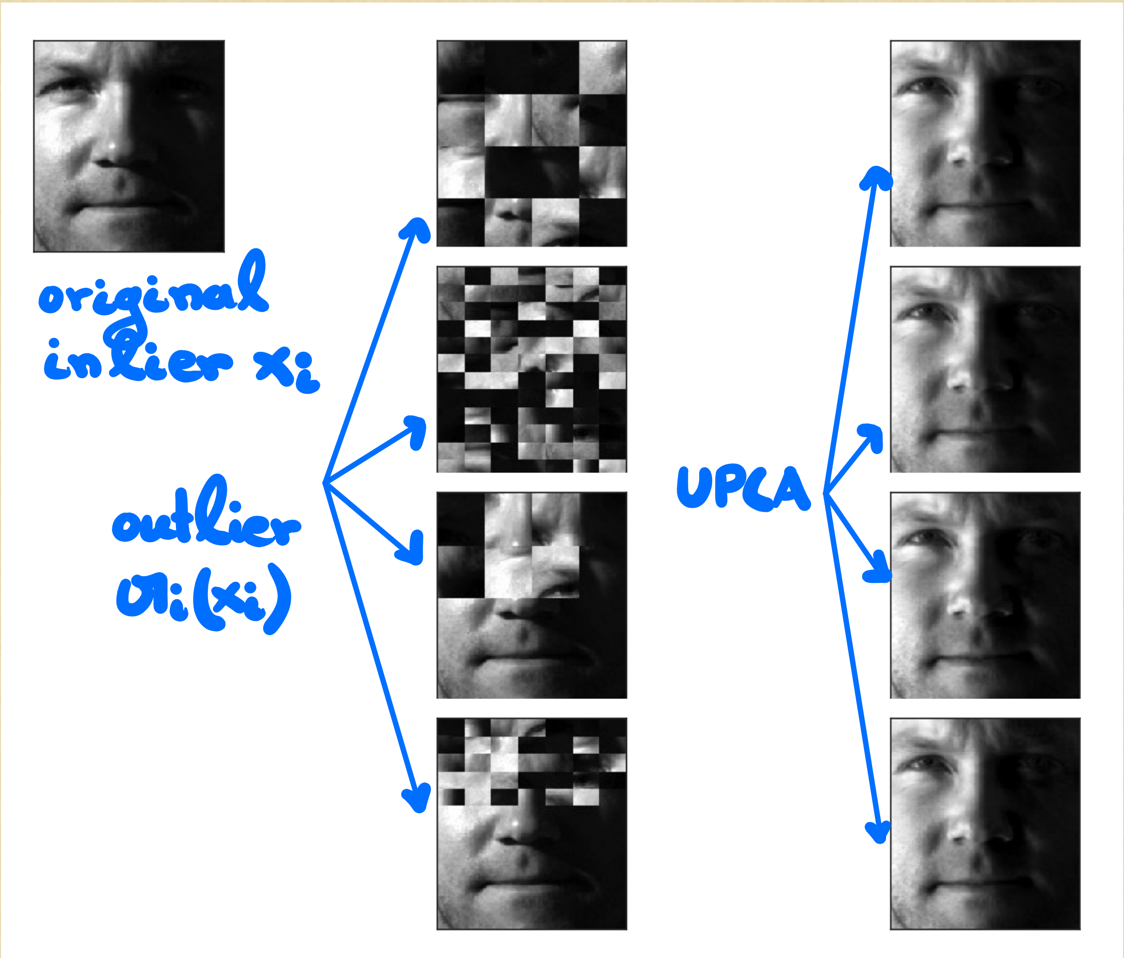
Ex 28



(a) r = 3 (b) r = 4 (c) r = 5

$\blacksquare$

Ex 29



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THANK YOU !