Birationally Equivalent Landau–Ginzburg models

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Objectives

- Toric potential
- Lie potential
- Birational equivalence of LG models

Landau–Ginzburg models

A Landau–Ginzburg model (LG) is a pair (X, w) formed by a variety X and a holomorphic function $w\colon X \to \mathbb{C}$ (or $\mathbb{P}^1$) called the superpotential.

Deformations

We consider both deformations of varieties and deformations of the potential, combining them to describe deformations of LG models, obtaining families

$$\mathrm{LG} \rightsquigarrow \mathrm{LG}'$$

where LG and LG′ behave very differently dualitywise.

LG models - examples

Especially well behaved LG models: Lefschetz fibrations!

Topological Lefschetz fibration

Let Y be a complex variety. A smooth function $f\colon Y \to \mathbb{C}$ (or $\mathbb{P}^1$) is a Topological Lefschetz fibration if:

- f has finitely many critical points of (holomorphic) Morse type so that around each critical point

$$f(z_0, \ldots, z_n) \simeq z_0^2 + \cdots + z_n^2.$$

- $f|_{Y - \{singular\ fibres\}}$ is locally trivial.

Symplectic Lefschetz fibrations

A a topological Lefschetz fibration $f\colon Y \to \mathbb{C}$ on a symplectic manifold (Y, ω) is a symplectic Letschetz fibration if:

- for every regular value $p \in \mathbb{C}$, the level Y_p is a symplectic submanifold of Y, and
- for each singular point Q_i the symplectic form ω_{Q_i} is non degenerate over the tangent cone of Y_{Q_i} at Q_i.

Examples from algebraic geometry

Donaldson proved that every symplectic 4 manifold has the structure of Lefschetz pencil.

A TLF can be obtained from blowing-up the base locus of the pencil.

TLFs from pencils

Modify a Lefschetz pencil by blowing up the base locus transforming it to a TLF.

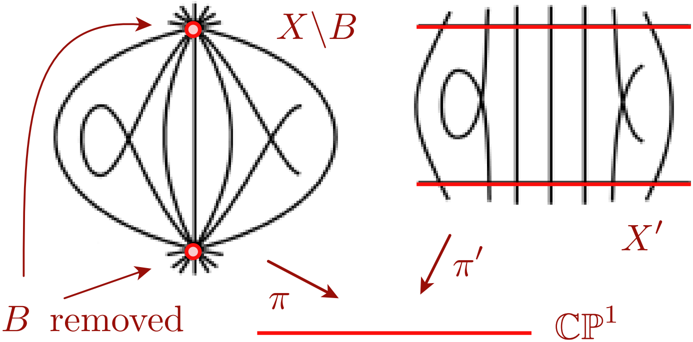

Figure: Pencil to fibration.

Symplectic Lefschetz fibrations via Lie theory

Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$ and Cartan subalgebra $\mathfrak{h}$. Given the Hermitian form $\mathcal{H}$ on $\mathfrak{g}$, define the symplectic form on $\mathfrak{g}$ by

$$\omega(X_1, X_2) = \operatorname{im} \mathcal{H}\left(X_1, X_2\right).$$

Symplectic Lefschetz fibrations via Lie theory

For $H_0 \in \mathfrak{h}$ we consider the adjoint orbit:

$$\mathcal{O}(H_0) = \mathrm{Ad}(G) \cdot H_0 = \{gH_0g^{-1} \in \mathfrak{g} : g \in G\},$$

together with the symplectic form ω.

Minimal adjoint orbit

Let $\mathcal{O}_n$ the adjoint orbit of $H_0 = \mathrm{Diag}(n, -1, \ldots, -1)$ in $\mathfrak{sl}(n+1, \mathbb{C})$, we call it the minimal adjoint orbit.

- The minimal adjoint orbit $\mathcal{O}_n$ is diffeomorphic to the cotangent bundle of the projective space $\mathbb{P}^n$.
- $\mathcal{O}_n$ is a nontoric Calabi–Yau manifold.

Lefschetz fibrations on adjoint orbits

Theorem (Gasparim, Grama, San Martin)

Given $H_0 \in \mathfrak{h}$ and $H \in \mathfrak{h}_{\mathbb{R}}$ with H a regular element.
The "height function" $f_H : (\mathcal{O}(H_0), \omega) \to \mathbb{C}$ defined by

$$\boxed{f_H(x) = \langle H, x \rangle \qquad x \in \mathcal{O}(H_0)}$$

has a finite number of isolated singularities and defines a symplectic Lefschetz fibration.

Minimal orbit of $\mathfrak{sl}(n+1,\mathbb{C})$

For

$$H_0 = \begin{pmatrix} n & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} \in \mathfrak{sl}(n+1,\mathbb{C}),$$

the orbit $\mathcal{O}(H_0)$ is diffeomorphic to $T^*\mathbb{P}^n$ and the potential

$$f_H(x) = \langle H, x \rangle, \quad x \in \mathcal{O}(H_0)$$

has $n+1$ critical points.

Example - Lie potencial on $\mathfrak{sl}(n+1,\mathbb{C})$

Let $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$ and choose H_0 such that the flag $\mathbb{F}_{H_0}$ is the projective space $\mathbb{P}^n$. We take

$$H_0 = \begin{pmatrix} n & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}, \qquad H = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{n+1} \end{pmatrix}.$$

Then

$$f_H\left(e^{\mathrm{ad}(Y)}e^{\mathrm{ad}(X)}H_0\right) =$$
$$= 2(n+1)\left[\mathrm{tr}(HH_0) + (\lambda_1-\lambda_2)\,x_1y_1 + \cdots + (\lambda_1-\lambda_{n+1})\,x_ny_n\right]$$

which is a degree 2 polynomial.

Local expression of the Lie potential

We call the expression of f_H written on a chart around H_0 the Lie potential on $\mathcal{O}_n \subset \mathfrak{sl}(n+1, \mathbb{C})$. It is given by

$$\boxed{\mathbf{f}_H(H_0) = \mathrm{tr}(HH_0) + (\lambda_1 - \lambda_2)x_1 y_1 + \cdots + (\lambda_1 - \lambda_{n+1})x_n y_n.}$$

HMS for the adjoint orbit $\mathcal{O}_1$ of $\mathfrak{sl}(2,\mathbb{C})$

Choose in $\mathfrak{sl}(2,\mathbb{C})$ the elements

$$H = H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Hence $\mathcal{O}_1$ is the set of matrices in $\mathfrak{sl}(2,\mathbb{C})$ with eigenvalues ± 1.

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- $\mathcal{O}_1$ forms a submanifold of $\mathfrak{sl}(2,\mathbb{C})$ of real dimension 4 (a complex surface).
- In this case the Weyl group is $\mathcal{W} = \{\pm 1\}$.
- Therefore, the potential $f_H =: \mathcal{O}_1 \to \mathbb{C}$ has two singularities, namely $\pm H_0$.

Fukaya–Seidel category for $\mathrm{LG}(2) = (\mathcal{O}_1, f_H)$

Lemma

The Fukaya–Seidel category $\mathit{Fuk}(\mathrm{LG}(2))$ *is generated by two Lagrangians* L_0 *and* L_1 *with morphisms:*

$$\mathrm{Hom}(L_i, L_j) \simeq \begin{cases} \mathbb{Z} \oplus \mathbb{Z}[-1] & i < j \\ \mathbb{Z} & i = j \\ 0 & i > j \end{cases} \tag{1}$$

where we think of $\mathbb{Z}$ *as a complex concentrated in degree* 0 *and* $\mathbb{Z}[-1]$ *as its shift, concentrated in degree* 1*, and the products* m_k *all vanish except for* $m_2(\cdot, \mathrm{I})$ *and* $m_2(\mathrm{I}, \cdot)$*.*

Quasi-Vampires

Theorem (Ballico, Barmeier, Gasparim, Grama, San Martin)

- $LG(2)$ *has no projective mirrors.*
- $\overline{LG(2)}$ *has no projective mirrors.*

This means:

For any projective variety X we have

$$D^b Coh(X) \neq Fuk(\mathrm{LG}(2)).$$

Potentials on $T^*\mathbb{P}^n$

Starting with a Hamiltonian action of $\mathbb{T} = \mathbb{C}^*$ on $T^*\mathbb{P}^n$ expressed in the open chart $V_0 = \{x_0 \neq 0\}$ by

$$\mathbb{T} \cdot V_0 = \left\{[1, t^{-1}x_1, \ldots, t^{-n}x_n], (ty_1, \ldots, t^n y_n)\right\}$$

we obtain a Hamiltonian vector field $T^*\mathbb{P}^n$

$$X(x_1, \ldots, x_n, y_1, \ldots, y_n) = (-x_1, \ldots, -nx_n, y_1, \ldots, ny_n).$$

and a potential

$$\mathbf{h}_c = \sum -2ix_iy_i + c.$$

Toric Potential

We call

$$\mathbf{h}_c = \sum -2ix_iy_i + c$$

a toric potential on $T^*\mathbb{P}^n$.

Mirrors from Linear data

Linear data associated to a toric Landau–Ginzburg model

- Div: encodes the divisor of the character-to-divisor map.
- Mon: describes infinitesimal action on monomials.

The dual toric Landau-Ginzburg model is obtained by exchanging Div and Mon, that is

$$\mathrm{Div}(X) = \mathrm{Mon}(f^\vee), \qquad \mathrm{Div}(X^\vee) = \mathrm{Mon}(f).$$

The Selfdual model LG_0

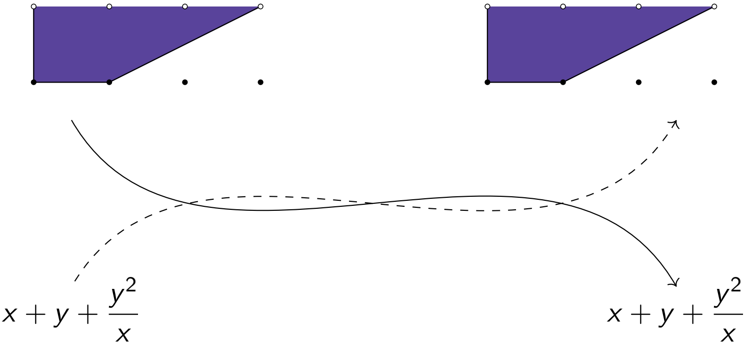
The LG model

$$\mathsf{LG}_0 = \left(T^*\mathbb{P}^1, x + y + \frac{y^2}{x} \right)$$

is dual to itself. Selfduality of this LG model is verified by simply pointing out that in this case the toric data is

$$\mathrm{Mon} = \mathrm{Div} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Example - Selfdual LG model



A nontrivial duality

Consider the Landau–Ginzburg models

$$(X, f) = \left(\mathbb{P}^2, x + y + \frac{1}{x} + \frac{1}{y}\right),$$

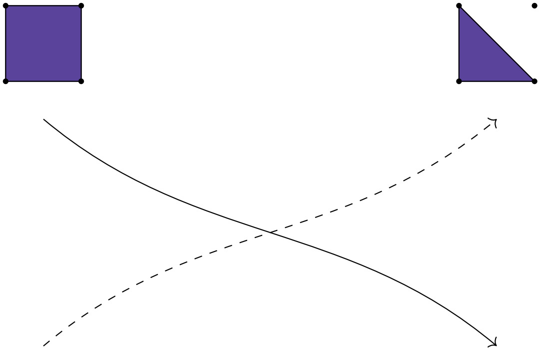
$$(Y, g) = \left(\mathbb{P}^1 \times \mathbb{P}^1, x + y + \frac{1}{xy}\right).$$

Since the Div matrix is given by the inward normals of the moment polytope, we have:

$$\mathrm{Div}_X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \mathrm{Div}_Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To see that (Y, g) is dual to (X, f) just observe that $\mathrm{Mon}_g = \mathrm{Div}_X$, and $\mathrm{Div}_Y = \mathrm{Mon}_f$.

A nontrivial duality



$$x + y + \frac{1}{xy} \qquad\qquad x + y + \frac{1}{x} + \frac{1}{y}$$

Deformation family

A (commutative) deformation family of a Landau–Ginzburg model (X, w) is a smooth family of Landau–Ginzburg models (X_t, w_t), with $t \in D \subset \mathbb{C}^n$ an open ball containing 0, such that $(X_0, w_0) = (X, w)$.

We call X_t a deformation of X_0, denoted by

$$(X_0, w_0) \rightsquigarrow (X_t, w_t).$$

Deformation of LG models - Example 1

Consider $T^*\mathbb{P}^1$ with coordinates $([1,x],y)$ and the Landau–Ginzburg models

$$\mathrm{LG}_0 = \left(T^*\mathbb{P}^1, x + y + \frac{y^2}{x}\right), \qquad \mathrm{LG}_1 = (T^*\mathbb{P}^1, 2x)$$

Using the potential

$$w_t = (1-t)2x + t\left(x + y + \frac{y^2}{x}\right)$$

on $X_t = T^*\mathbb{P}^1$ we obtain the deformation

$$\mathrm{LG}_0 \rightsquigarrow \mathrm{LG}_1 \,.$$

Deformation of LG models - Example 2

Our next objective is to describe how duality works for the deformation family

$$\mathsf{LG}_1 = (T^*\mathbb{P}^1, h_c) \rightsquigarrow \mathsf{LG}_2 = (\mathcal{O}_1, f_H).$$

We will use the deformation of the Hirzebruch surfaces $\mathbb{F}_2$ to $\mathbb{F}_0$, extending it to a deformation of partially compactified Landau–Ginzburg models.

$$(\mathbb{F}_2, \mathbf{h}) \rightsquigarrow (\mathbb{F}_{\mathbf{0}}, \mathbf{f}).$$

Deformation of LG models - Example 2

We will make use of the following result:

Lemma

$\mathbb{F}_2$ *deforms to* $\mathbb{F}_0$.

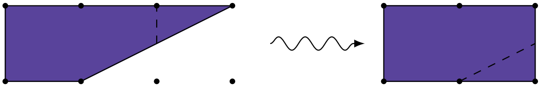
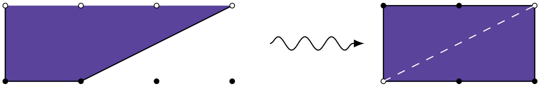


Figure: $\mathbb{F}_2$ deforms to $\mathbb{F}_0$

Deformation of LG models - Example 2

This induces the deformation $T^*\mathbb{P}^1 \rightsquigarrow \mathcal{O}_1$.

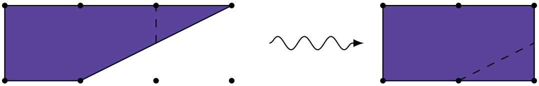


Deformation of LG models - Example 2

The deformation of LG models:

$$(T^*\mathbb{P}^1, h) \rightsquigarrow (\mathcal{O}_1, f_H)$$

is obtained from the deformation $\mathbb{F}_2$ to $\mathbb{F}_0$.



Deformation of LG models - Example 3

Combining the deformations

$$\mathsf{LG}_0 \rightsquigarrow \mathsf{LG}_1, \qquad \mathsf{LG}_1 \rightsquigarrow \mathsf{LG}_2,$$

we obtain a new deformation that changes both the variety and the potential, namely

$$\mathsf{LG}_0 \rightsquigarrow \mathsf{LG}_2 \, .$$

We then wish to compare the mirrors of LG_0 and LG_2, and we will see that they behave very differently. Since $T^*\mathbb{P}^1$ is a toric variety, we can use toric duality for LG_0.

Comparing the Lie potential and the toric potential

On the open chart $V_0 = \{x_0 \neq 0\}$ of $T^*\mathbb{P}^n$ we have defined two potentials:

Toric potential:

$$\mathbf{h}_c = -2x_1y_1 - \ldots - 2nx_ny_n + c$$

Lie potential:

$$\mathbf{f}_H = < \cdot, H >$$

Birationally equivalent LG models

Theorem (S., 2023)

For each $n \in \mathbb{N}$, there exist matrices $H, H_0 \in \mathfrak{sl}(n+1, \mathbb{C})$ and a constant $c \in \mathbb{C}$ such that the Lie potential on the minimal adjoint orbit $\mathcal{O}_n$ and the toric potential on the cotangent bundle $T^*\mathbb{P}^n$ coincide on dense open charts, that is $\mathbf{f}_H = \mathbf{h}_c$.

Proof

Take

$$H_0 = \begin{pmatrix} n & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} \in \mathfrak{sl}\left(n+1, \mathbb{C}\right),$$

$$H = \begin{cases} \mathrm{Diag}\left(-n, \ldots, -4, -2, 0, 2, 4, \ldots, n\right) & \text{if } n \text{ is even,} \\ \mathrm{Diag}\left(-n, \ldots, -3, -1, 1, 3, \ldots, n\right) & \text{if } n \text{ is odd} \end{cases}$$

and $c = -n^2 - n$.

Proof (cont.)

On the Lie side, we have:

$$\mathbf{f}_H(H_0) = \mathrm{tr}(HH_0) + (\lambda_1 - \lambda_2)x_1y_1 + \cdots + (\lambda_1 - \lambda_{n+1})x_ny_n,$$

where the eigenvalues of $H = \mathrm{Diag}(\lambda_1, \ldots, \lambda_{n+1})$ satisfy

$$\lambda_1 - \lambda_j = -2(j-1),$$

so that

$$\mathbf{f}_H(H_0) = -n^2 - n - 2x_1y_1 - \cdots - 2nx_ny_n.$$

On the toric side,

$$\begin{aligned} \mathbf{h}_c &= c - 2x_1y_1 - \ldots - 2nx_ny_n \\ &= -n^2 - n - 2x_1y_1 - \cdots - 2nx_ny_n. \end{aligned}$$

Birational equivalence

Two Landau–Ginzburg models are called birationally equivalent if their domains are birationally equivalent varieties and their potentials coincide on Zariski open sets.

Birational Equivalence

A rigorous version of our result may be stated as:

Theorem (S., 2023)

For each $n \in \mathbb{N}$*, there exist matrices* $H, H_0 \in \mathfrak{sl}(n+1, \mathbb{C})$ *and a constant* $c \in \mathbb{C}$ *such that the LG models* $(\mathcal{O}_n, f_H)$ *and* $(T^*\mathbb{P}^n, h_c)$ *are birationally equivalent.*

Proof.

The expressions of $\mathbf{f}_H$ and $\mathbf{h}_c$ for the potentials f_H and h_c coincide on Zariski open sets. □

Example - Duality vs. deformation

We consider $T^*\mathbb{P}^1$ with coordinates $([1,x],y)$ and the family of potentials

$$w_t = (1-t)\left(x + y + \frac{y^2}{x}\right) + t(2x).$$

Then we have that the initial LG model LG_0 is selfdual, while the final LG model is defined by $\mathrm{LG}_1 = (T^*\mathbb{P}^1, 2x)$ which has the toric data

$$\mathrm{Div} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \mathrm{Mon} = (1\,0).$$

Example - Duality vs. deformation

Now, inverting the matrices by toric duality gives us the $\mathsf{LG}_1^\vee$ model

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Example - Duality vs. deformation

Now, inverting the matrices by toric duality gives us the $\mathsf{LG}_1^\vee$ model

$$\mathsf{Div} = (1\,0) \quad \mathsf{Mon} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \end{pmatrix}.$$

We conclude that this family takes a selfdual LG model to another very far from selfdual.

$$\begin{array}{ccc} \mathsf{LG}_0 & \to & \mathsf{LG}_1 \\ \| & & \not\sim \\ \mathsf{LG}_0^\vee & \to & \mathsf{LG}_1^\vee \end{array}.$$

The End

Obrigado! $\rightsquigarrow$ Thank you!