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Gromov-Witten Theory of Non-Convex Complete Intersections

contains results joint with

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+

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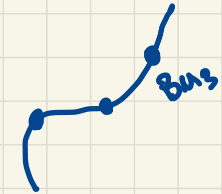
$\boxed{\text{Orbifold GW theory}}$



- $\overline{\mathcal{M}}_{g,n}(X,\beta) = \{ f : C \longrightarrow X \mid f \text{ stable, representable}, f_*[C]=\beta \}/\sim$

  DM stack!

  genus g, n-marked, only stacky at marks

- $ev_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow IX$ evaluation morphism

  inertia stack

- $H^*_{CR}(X)$ Chen-Ruan cohomology

  $H^*_{CR}(X) = H^*(IX)$ as non-graded groups.

  ring structure + grading more complicated

- GW-invariant

$$\langle \gamma_1, \dots, \gamma_n \rangle_{g,n,\beta} = \int_{[\overline{\mathcal{M}}(X,\beta)]^{vir}} \prod_i ev_i^* \gamma_i \qquad \gamma_i \in H^*_{CR}(X)$$

$\boxed{\text{Quantum Lefshetz}}$

Assume $g=0$

Let

$$\begin{tikzcd} & E = \bigoplus_i L_i \arrow[d] \\ V(s) = X \arrow[r, hook, "i"] & Y \arrow[u, bend right, "s"'] \end{tikzcd}$$

__Thm__

Assume E in convex

i.e. $H^1(C, f^*E) = 0$
for all stable maps
$C \xrightarrow{f} Y$

Then

$$i_*[M_{0,n}(X,\beta)]^{vir} = [M_{0,n}(Y, i_*\beta)]^{vir} \cap e(E_{0,n,\beta})$$

$\rightsquigarrow$ GW invariants of X can be computed in terms of GW of Y

When X is a scheme

easier to check

$$E \text{ convex} \longleftrightarrow c_1(L_i)\cdot\beta \geq 0 \quad \forall i$$

e.g. for $Y = \mathbb{P}^n$, $L_i = \mathcal{O}(n_i)$

convex if $n_i \geq 0 \quad \forall i$

But this is <u>not</u> true when X is an orbifold: (CGIJJM, 12)

E convex $\longleftrightarrow$ pulled back from coarse moduli space

very restrictive assumption

e.g. $Y = \mathbb{P}(w_0, \ldots, w_n)$, $L_i = \mathcal{O}(n_i)$

need n_i divisible by $\gcd(w_0, \ldots, w_n)$

$Q_5 \subset \mathbb{P}^4$
$g = 0$

$\therefore$ Scheme-theoretic proofs will not work in orbifold case

Goal: Find a way to compute invariants for complete intersections when convexity fails

Assume for talk: X is Calabi-Yau threefold in weighted projective stack $\mathbb{P}(w_0, \ldots, w_n)$

e.g.

$$\begin{tikzcd} & \mathcal{O}(\textstyle\sum_i w_i) \arrow[d, shift right] \\ X \arrow[r, hook] & \mathbb{P}(w_0, \ldots, w_n) \arrow[u, shift right, "s"'] \end{tikzcd}$$

Quasimap Theory

Consider
- W affine variety
- G reductive group acting on W
- θ G-character

Don't need lci assumption!

Have two stack quotients

$$[W/\!\!/_\theta G] \subset [W/G]$$

Artin stack

Def. A quasimap to $X = [W/\!\!/_\theta G]$ is a representable morphism

$$f : C \longrightarrow [W/G]$$

s.t. $f^{-1}([W/G] - [W/\!\!/_\theta G])$ is zero-dim and contains no markings

$\exists$ a family of stability conditions,

parameterized by $\varepsilon \in \mathbb{Q}_{>0} \cup \{0^+, \infty\}$

$\rightsquigarrow$ get family of moduli spaces $Q^{\varepsilon}_{0,n}(X,\beta)$

- $Q^{\infty}_{0,n}(X,\beta) = \overline{\mathcal{M}}_{0,n}(X,\beta)$ moduli of stable maps

Can "wall-cross"

$$Q^{0^+}_{0,n}(X,\beta) \xleftrightarrow{\ \varepsilon\ } Q^{\infty}_{0,n}(X,\beta)$$

easier to work with

on $\varepsilon = \infty$ side, define

ϕ_i, ϕ^i cohom classes that are Poincaré duality

$$J(t,q,z) = 1 + t/z + \sum_{n,\beta} q^\beta \phi_i \left\langle \frac{\phi^i}{z - \psi_1}, t, \dots, t \right\rangle_{0,n,\beta}$$

$t \in H^*_{CR}(X)$ generic element, q, z formal variables

generating series of GW invariants with insertions t

on $\varepsilon = 0^+$, have a series

$$I(q,z)$$

defined by localization on substack of $\mathrm{Hom}(\mathbb{P}(1,r), [W/G])$

$$\mathbb{C}^* \curvearrowright \mathbb{P}(1,r) \qquad \lambda \cdot [x:y] = [x:\lambda y]$$

Can be computed explicitly

Thm (Zhou) The two series satisfy

$$J(\underbrace{\mu(q,z)}_{t}, q, z) = I(q,z)$$

where $\mu(q,z) = [zI - z]_+$ non-negative part

$$I = 1 + O(z^{-1})$$

Note: Generic insertion t of J depends on I here

Questions:

1. What type of insertions can we obtain?

2. Can we recover individual GW invariants?

Admissible Classes

We define a subring of $\mathcal{H} \subset H^*_{CR}(X)$ that we call the <u>admissible state space</u>. Call $\phi \in \mathcal{H}$ <u>admissible class</u>.

Notably, we have

$$H^*_{amb}(X) \subset \mathcal{H} \subset H^*_{CR}(X)$$

inclusions can be strict

Subring generated by classes pulled back from ambient space

Classes Poincaré dual to cycles defined by coordinate hyperplane vanishings

Extended GIT

Use that I-function is sensitive to GIT presentation but J is not

After specifying a certain basis $\{\phi_1, \dots, \phi_m\}$ of admissible classes,

introduce new GIT presentation

$$X = \left[W_e /\!/_{\theta_e} (\mathbb{C}^*)^{m+1} \right]$$

$W_e \subset \mathbb{C}^{n+m+1}$
affine scheme

extra torus factor
for each class ϕ_i

for weighted proj space, take $\theta_e = (1, \dots, 1)$

- The weight matrix of the action by $(\mathbb{C}^*)^{m+1}$ looks like

$$\left(\begin{array}{ccc|ccc} w_0 & \cdots & w_n & 0 & \cdots & 0 \\ \hline a_{10} & \cdots & a_{0n} & 1 & & 0 \\ \vdots & & \vdots & & \ddots & \\ a_{m0} & \cdots & a_{mn} & 0 & & 1 \end{array}\right)$$

weights a_{ij} are explicitly defined depending on class ϕ_i in chosen basis

- We defined by extending the defining vector bundle and section to $\mathbb{C}^{n+m+1}$
  - Explicit choice of extension based on weight matrix

Main Result

There is an explicit I function associated to the extended GIT

multiple Novikov parameters

$$I(q_0, \dots, q_m, z) \in \mathcal{H}[z][\![q_0, \dots, q_m]\!]$$

admissible classes

$\downarrow$

$\sum t_i \phi_i$

and an invertible ring homomorphism

$$\mathbb{Q}[\![Q, t_1, \dots, t_m]\!] \rightleftarrows \mathbb{Q}[\![q_0, \dots, q_m]\!]$$

s.t.

$$J\left(\textstyle\sum t_i \phi_i, Q, z\right) \rightleftarrows I(q_0, \dots, q_m, z)$$

generic insertion in $\mathcal{H}$

$\mathcal{H}$ contain it

Some remarks:

- invertibility ensures recovery of individual GW invariants by giving explicit formula for $J(t, Q, z)$

- By our choice of ϕ_i, all admissible classes appear as insertions. This includes all invariants normally computed by a Quantum-Lefschetz type theorem

Example.

$$\begin{array}{c} \mathcal{O}(7) \\ \downarrow \\ X_7 \subset \mathbb{P}(1,1,1,1,3) \end{array} \to \mathbb{P}^4$$

degree 3

Possible equation: $x_0^7 + x_1^7 + x_2^7 + x_3^7 + x_4^2 x_3 = F$

$$V(F) = W \rightsquigarrow X_7 = [W /\!\!/_\theta \mathbb{C}^*]$$

$$[*/\mu_3] = B\mu_3$$

$$IX_7 = X_7 \sqcup B\mu_3 \sqcup B\mu_3$$

degree 2 class corresponding to $B\mu_3$ sector

$$H^*_{CR}(X) = H^*(X) \oplus \langle \phi_{1/3} \rangle \oplus \langle \phi_{2/3} \rangle$$

degree 4 class

Recall I-function computed from quasimaps $\mathbb{P}^1(1,r) \longrightarrow [W/\mathbb{C}^*]$

only one stacky point.

- However, convexity condition only starts failing for curves with >1 stacky point

$\rightsquigarrow$ Naive I-function doesn't capture information of invariants involved in convexity failure (i.e. those invariants with multiple insertions)

Problem:

To get more expressive I-function, need to encode data of multiple stacky points into quasimaps from $\mathbb{P}(1,r)$

" Extended GIT idea "

$\mathbb{P}(1,1,1,1,3)$

$$\begin{tikzcd} \text{"}p/3\text{"} \arrow[r, mapsto] & \mathcal{B}\mu_3 \ni \mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"] & {[W/\mathbb{C}^*]} \\ & p \in \underline{C} \arrow[ur, dotted, "\exists ?"] & \end{tikzcd}$$

coarse moduli map

$$f \longleftrightarrow \left\{ \begin{array}{ll} \mathcal{L} & s_0, \dots, s_3 \in H^0(\mathcal{L}) \\ \downarrow & s_4 \in H^0(\mathcal{L}^3) \\ \mathcal{C} & \end{array} \right\}$$

root bundle. T "=" $\mathcal{O}(p/3)$, $T^{\otimes 3} \cong \mathcal{O}(p)$

$\mathcal{L} = \pi^* L \otimes T^i$ for some L, i

$H^\bullet(\mathcal{L}) \cong H^\bullet(L)$ but $H^\bullet(\mathcal{L}^{\otimes 3}) \cong H^\bullet(L^3 \otimes \mathcal{O}(p))$

Defining quasimap $\underline{C} \to [W/\mathbb{C}^*]$ using L misses data

Write

$$X = [W_e /\!/_\theta (\mathbb{C}^*)^2]$$

Ex $W_e = V((x_1^7 + \cdots + x_3^7)y^2 + x_3 x_4^2)$

degree $(7,2)$

$(\mathbb{C}^*)^2$ acts on $\mathbb{C}^6$ with weight matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

With this presentation, quasimap to X requires 2 line bundle choices

Consider

$\underline{C} \longrightarrow [W_e/(\mathbb{C}^*)^2]$ defined by

bundles : $L, \mathcal{O}(p)$

sections s_i determined by $H^0(\mathcal{L}) \cong H^0(L)$, $H^0(\mathcal{L}^3) = H^0(L \otimes \mathcal{O}(p))$

last section s_6 is tautological section of $\mathcal{O}(p)$

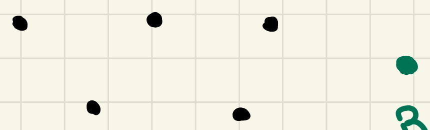
$\Rightarrow$ get quasimap from $\underline{C}$ that agrees with original from $\mathcal{C}$ ← replaces orbipoints with basepoints

Example of non-ambient admissible class

$X_{24} \subset \mathbb{P}(1,4,4,6,9)$

(generically)

IX_{24} contains sector that looks like



$B\mu_3$ stacks

$B\mu_9$

$\rightsquigarrow$ generate $\mathbb{C}^6$ in $H^2_{CR}(X)$

$\mathbb{P}(6,9)$

ambient sector

Poincaré dual of this class is admissible but not ambient

Results hold for general complete intersections in Toric Stacks

i.e. $X \subset [V /\!\!/_{\theta} T]$ for T a torus

Biggest Change: Complexity of mirror map depends

depends on degrees of classes you extend by

e.g. after extending by classes of high degree, mirror map for Fano hypersurface may resemble that of general type hypersurface

Non-Abelian Quotients

Want to consider $X = [W // G]$ $\quad G$ not abelian

Assume G is connected

Thm (Webb)

Let $T \subset G$ be maximal torus

Then

$I_{[W/G]}$ is obtained by modifying $I_{[W/T]}$ with an abelianization factor

explicitly computable

Let W be the Weyl group of $T \subset G$

moves sectors around

$\rightsquigarrow$ W acts on $H^*_{CR}(X_T)$, induced by action on IX_T

abelian quotient

Weyl-invariant classes give classes in $H^*_{CR}(X_G)$ non-abelian quotient

<u>Thm</u> (S. Webb)

Suppose γ is the fundamental class of a Weyl-invariant twisted sector

Then there exists a GIT extension and extended I-function that captures data about GW invariants with insertions γ

Application Del Pezzo in weighted Grassmannian

$$X_{1,7/3} \subset w\mathrm{Gr}(2,5)$$

$$w\mathrm{Gr}(2,5) : \quad \mathbb{C}^{10} /\!\!/ Gl_2 .$$

$B\mu_3 \subset w\mathrm{Gr}(2,5)$

For $Gl_2 = (Sl_2 \times \mathbb{C}^*)/\mu_2$, action given by

$$(\lambda, \mu) \cdot \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{bmatrix} = \lambda \cdot \begin{bmatrix} \mu a_1 & \mu a_2 & \mu a_3 & \mu^3 a_4 & \mu^3 a_5 \\ \mu b_1 & \mu b_2 & \mu b_3 & \mu^3 b_4 & \mu^3 b_5 \end{bmatrix}$$

$X_{1,7/3}$ defined by generic section $s \in L^{\oplus 4}$

L defined by character $(\lambda, \mu) \longrightarrow \mu^4$

Oneto-Petracci give a conjectured formula for a specialization of the Quantum Period of $X_{1,7/3}$

Defined as specialization of $J(t, Q, z)$ along unit class, where $t = \sum t_i \mathbb{1}_{g_i}$

$\mathbb{1}_{g_i}$ is unit class of twisted sector s.t. $\deg(\mathbb{1}_{g_i}) < 2$.

For $X_{1,7/3}$, there is one such class $\mathbb{1}_{1/3}$ that is required to obtain quantum period and it is Weyl-invariant!

① We can compute I-function of GIT extended by 1_{Y_3}

abelian-non-abelian corr. used.

② Can obtain a formula for $J(t \cdot 1_{Y_3}, Q, z)$ explicitly from I

③ Specialize to recover full quantum period

Thm

We show that the quantum period obtained above agrees with the conjectured formula after an explicit specialization

OP's conjecture part of a larger conjecture on orbifold Del Pezzos

Imprecisely stated ...

Conj (Coates, Kasprzyk, ..)

Regularized Quantum Period = Classical period of Laurent polynomial obtained via toric degeneration

computed by S-Webb

computed via program by Coates-Kasprzyk

$\overset{?}{=}$

Computational evidence suggests yes.

(Future work)