Smooth Hilbert Schemes

Gregory G. Smith

Queen's University

CHALLENGE: Understand all varieties that embed into $\mathbb{P}^n$ (over a field $k = \bar{k}$).

GROTHENDIECK (1961): Closed subschemes in $\mathbb{P}^n$ are parametrized by $\mathrm{Hilb}(\mathbb{P}^n)$:

- $X \subset \mathbb{P}^n$ corresponds to a point $\mathrm{Hilb}(\mathbb{P}^n)$
- deformations correspond to nearby points
- one-parameter families correspond to curve.

BASIC PROBLEM: Describe the geometry of the projective scheme $\mathrm{Hilb}(\mathbb{P}^n)$.

HILBERT (1890): There exists $P_X \in \mathbb{Q}[t]$ such that $P_X(j) = \dim_k(k[x_0, x_1, \ldots, x_n]/I_X)_j$ for all $j \gg 0$.

EXAMPLE: The twisted cubic curve is the image of the map $\mathbb{P}^1 \to \mathbb{P}^3$ given by

$$[z_0 : z_1] \longmapsto [z_0^3 : z_0^2 z_1 : z_0 z_1^2 : z_1^3]$$

and $I_C = \langle x_1x_3 - x_2^2, x_0x_3 - x_1x_2, x_0x_2 - x_1^2 \rangle$, so

$$P_C(t) = 3t + 1 = \binom{t+1}{1} + \binom{t}{1} + \binom{t-1}{1} + \binom{t-3}{0}.$$

GROTHENDIECK (1961): We have this disjoint union

$$\mathrm{Hilb}(\mathbb{P}^n) = \coprod_{q \in \mathbb{Q}[t]} \mathrm{Hilb}^q(\mathbb{P}^n).$$

MACAULAY(1926): $\mathrm{Hilb}^q(\mathbb{P}^n) \neq \emptyset$ if and only if there exists $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \in \mathbb{N}^r$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant 1$ and $q(t) = \sum_{i=1}^{r} \binom{t+\lambda_i - i}{\lambda_i - 1}$

HARTSHORNE(1966): Every nonempty $\mathrm{Hilb}^q(\mathbb{P}^n)$ is path-connected.

REEVES-STILLMAN(1997): Every nonempty $\mathrm{Hilb}^q(\mathbb{P}^n)$ has a smooth point.

$\boxed{\text{GOAL: Classify all smooth } \mathrm{Hilb}^q(\mathbb{P}^n).}$

MUMFORD (1962): An irreducible component of $\mathrm{Hilb}^{14t-23}(\mathbb{P}^3)$ is generically nonreduced. (singular at every point).

ELLIA-HIRSCHOWITZ-MEZZETTI (1992): The number of components of $\mathrm{Hilb}^{dt+1-g}(\mathbb{P}^3)$ is not bounded by a polynomial in d and g.

VAKIL (2006): Every singularity type appears in some $\mathrm{Hilb}^{q}(\mathbb{P}^4)$.

Smooth Examples:

- $X \subset \mathbb{P}^n$ is a $(\lambda_1 - 1)$-dimensional linear subspace if and only if $q(t) = \binom{t+\lambda_1-1}{\lambda_1-1}$, so $r = 1 \Rightarrow$

$$\mathrm{Hilb}^q(\mathbb{P}^n) = \mathrm{Gr}(\lambda_1 - 1, \mathbb{P}^n).$$

- $X \subset \mathbb{P}^n$ is a hypersurface of degree r if and only if $q(t) = \sum_{i=1}^{r} \binom{t+n-i}{n-1}$, so $\lambda = (n^r) = (n, n, \dots, n) \Rightarrow$

$$\mathrm{Hilb}^q(\mathbb{P}^n) = \mathbb{P}^{\binom{r+n}{n}-1}.$$

- For all λ such that $\lambda_1 > \lambda_2 > \dots > \lambda_r > 1$, $\mathrm{Hilb}^q(\mathbb{P}^n)$ is simply a partial flag variety

THEOREM (Skjelnes-Smith): $\mathrm{Hilb}^{q}(\mathbb{P}^n)$ is smooth if and only if one of following holds:

(0) $n \leq 2$,

(1) $\lambda_r \geq 2$,

(2) $\lambda = (1)$ or $\lambda = (n^{r-2}, \lambda_{r-1}, 1)$ where $r \geq 2$ and $n \geq \lambda_{r-1} \geq 1$

(3) $\lambda = (n^{r-s-3}, \lambda_{r-s-2}^{s+2}, 1)$ where $r-3 \geq s \geq 0$ and $n-1 \geq \lambda_{r-s-2} \geq 3$,

(4) $\lambda = (n^{r-s-5}, 2^{s+4}, 1)$ where $r-5 \geq s \geq 0$,

(5) $\lambda = (n^{r-3}, 1^3)$ where $r \geq 3$.

Fogarty ('68)

Ramkumar ('19)

Staal ('20)

DEFINITIONS: An inclusion $Y \subset X$ is residual if there exists a linear subspace $\Lambda \subseteq \mathbb{P}^n$ and a hypersurface $D \subset \Lambda$ such that $D \subseteq X \subseteq \Lambda$ and $\mathcal{I}_X = \mathcal{I}_Y \cdot \mathcal{I}_D$ $(X = Y \cup D)$. A residual flag is a chain $\emptyset \subset X_\ell \subset X_{\ell-1} \subset \cdots \subset X_1$ of such inclusions

THEOREM: Points is a smooth $\mathrm{Hilb}^q(\mathbb{P}^n)$ satisfying (1)-(5) correspond to either a residual flag or the union of a residual flag and a point.