Toward the logarithmic Hilbert scheme

Online Algebraic Geometry Seminar 03/23

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Question: What is a good refinement of coherent sheaves in log geometry?

Basic cases:
- $D \subset X$ nc divisor
- $(X, X_0) \to (S, 0)$ nc degeneration

} $\rightarrow$ toroidal / log smooth

Applications:
- log DT/PT for pairs, toric degen's $\rightarrow$ gluing formulae, computations
- induced degenerations of moduli spaces of sheaves
- " " of Hyperkähler $\mathrm{Hilb}^n K3$
- constructions of bundles on X by deformation from X_0
- constructions of stability conditions " "

Candidates:

I) Parabolic sheaves (Borne/Vistoli/Talpo 2010/14)

coherent sheaves on infinite root stack associated to $(X, \mathcal{M}_X)$

II) Transverse sheaves on expanded degenerations (J. Li/B. Wu 2011: D smooth; Maulik/Ranganathan 2020: 1d supp $\downarrow$ log DT)

III) $\mathcal{O}_X^{\log} = \mathcal{O}_X[\mathcal{M}_X]/\mathcal{O}_X^{\times}$ - modules (..., STT 2019-2021)

IV) $\boxed{\text{Sheaves on "closed log subschemes"}}$ (STT 2021-, Kennedy-Hunt 2023)

this talk: restrict to <u>log Hilbert scheme</u>

Logarithmic geometry (via stacks)

$\sigma \subset \mathbb{R}^n$ rational polyhedral cone $\rightsquigarrow$ $A_\sigma = \operatorname{Spec} \mathbb{Z}[\sigma^\vee \cap \mathbb{Z}^n]$ affine toric variety

$\circlearrowleft$

$T_\sigma = \operatorname{Spec} \mathbb{Z}[\mathbb{Z}^n]$

Stack quotient: $\mathcal{A}_\sigma = [A_\sigma / T_\sigma]$

Expl: $[\mathbb{A}^2 / \mathbb{G}_m^2]$

$\rightarrow$ 4 strata $B\mathbb{G}_m \; \bullet \quad \bullet \; \mathrm{pt}$ (open)

(closed) $B\mathbb{G}_m^2 \; \bullet \quad \bullet \; B\mathbb{G}_m$

$$\begin{tikzcd} U \arrow[r] \arrow[d] & A_\sigma \\ X & \end{tikzcd}$$

Log structure on X:

$X \to \mathcal{L}og = \varinjlim_{\sigma} \mathcal{A}_\sigma$ [Olsson]

$\rightsquigarrow$ strata on X: pull-back from $\mathcal{A}_\sigma$ locally decorated by cones σ

Tropicalization:

$\operatorname{Trop} X = \varinjlim_{\sigma} \sigma$ cone complex

Expls: • log points $\operatorname{Spec} k \to \mathcal{A}_\sigma$ $\quad \hookleftarrow \operatorname{Spec} k \hookrightarrow A_\sigma$

• $D = D_1 \cup \dots \cup D_r \subseteq X$ snc $\rightsquigarrow$

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\text{étale}"'] \arrow[dr, dashed] & \mathbb{A}^r \arrow[d] \\ X \arrow[r] & \mathcal{A}_{\mathbb{R}^r_{\geq 0}} \end{tikzcd}$$

these patch

[or just (X,D) toroidal]

• pull-back: $X \to \mathcal{L}og$, $Y \to X$ $\rightsquigarrow$ $Y \to X \to \mathcal{L}og$

• log modification: $\tilde{X} \to X$ s.th. locally in Y

$\Sigma \to \sigma$ subdivision

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{A}_\Sigma = \varinjlim_{\tau \in \Sigma} \mathcal{A}_\tau \arrow[d] & TV(\Sigma) \arrow[l] \arrow[d] \\ X \arrow[r] & \mathcal{A}_\sigma & A_\sigma \arrow[l] \end{tikzcd}$$

i.e. locally given by a toric birational morphism

I. A failed approach

$\mathcal{M}_{A_\sigma} = \mathcal{O}^\times_{A_\sigma \setminus D_\sigma} \cap \mathcal{O}_{A_\sigma} \hookrightarrow \mathcal{O}_{A_\sigma}$

$\alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\simeq} \mathcal{O}_X^\times$

$\cap \qquad \cap$

$\alpha : \mathcal{M}_X \to \mathcal{O}_X$

Tempting: Consider ideals in the "log structure sheaf"

$$\mathcal{O}_X^{\log} = \mathcal{O}_X[\mathcal{M}_X]/\mathcal{O}_X^\times \qquad h \cdot z^m \sim z^{h \cdot m} \text{ for } h \in \mathcal{O}_X^\times.$$

Pblm: $\mathcal{O}_X^{\log}$ has <u>too many ideals</u> to be useful wholesale

One can nevertheless restrict stalkwise to prove a correspondence [STT 2021]

$$\left\{ I^{\log} \subseteq \mathcal{O}^{\log}_{X \times S, pt} \right\} \longleftrightarrow \text{transverse subschemes } Z \hookrightarrow \tilde{X}_S$$

$\tilde{X}_S \downarrow X \times S$ — toric blowing up

But: no good global theory.

II. LogHilb: Closed log subschemes

For simplicity: target X rather than X/B

Def: closed log embedding of X over S:

$$\begin{tikzcd} & & \tilde{X}_S \arrow[d] \\ Z \arrow[urr, dashed, "\text{strict}"] \arrow[rr, "f"] \arrow[drr, "\pi"'] & & X_S = X \times S \arrow[d] \\ & & S \end{tikzcd}$$

- π log flat
- f factors $Z \xrightarrow{\imath} \tilde{X}_S \xrightarrow{\varkappa} X_S$ ($\varkappa$: log modification)

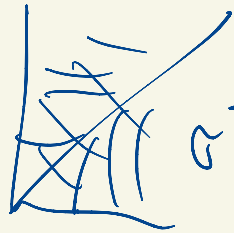
  ($\imath$:) strict closed embedding, i.e. log str. locally on Z pulled back from $\tilde{X}_S$

Note: Is relative even if S is a pt: $S = \mathrm{Spec}(Q \to k)$
$= \mathrm{Spec}(k \to A_{\sigma = Q_{\mathbb{R}}^\vee})$

Log modification: locally in X_S

$$\begin{tikzcd} \tilde{X}_S \arrow[r] \arrow[d] & TV(\Sigma) \arrow[d] \\ X_S \arrow[r, "\text{chart}"] & A_\sigma = \mathrm{Spec}\, \mathbb{Z}[\sigma^\vee \cap \mathbb{Z}^n] \end{tikzcd}$$

Σ subdivision of σ



$\sigma^\vee$

Log flatness (K. Kato, Olsson, Gillam, Ogus; Tevelev)

over log pt, locally on Z:

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "f"'] & A_P \arrow[d, "\pi"] \\ \mathrm{Spec}(Q \to k) \arrow[r] & A_Q \end{tikzcd}$$

f log flat $\iff$ $\underline{Z} \times G(P^{gp}/Q^{gp}) \to \pi^{-1}(0)$ flat

Expl: a) $Q=0$ Z/k log flat $\iff$ $Z \times G(P^{gp}) \to A_P$ flat (cf. Tevelev)

i.e. log flatness means toric transversality

b) Wu/Li: $D \subset X$ smooth divisor $P=\mathbb{N}$, $Q=0$ $\mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_D) = 0$

c) (i) $Z = (\mathbb{A}^1, \mathrm{fil}^{*}_{\mathbb{A}^2}) \xrightarrow{f} X = (\mathbb{A}^2, V(zw))$

$t \longmapsto (t,t)$

Z/k not log-flat:

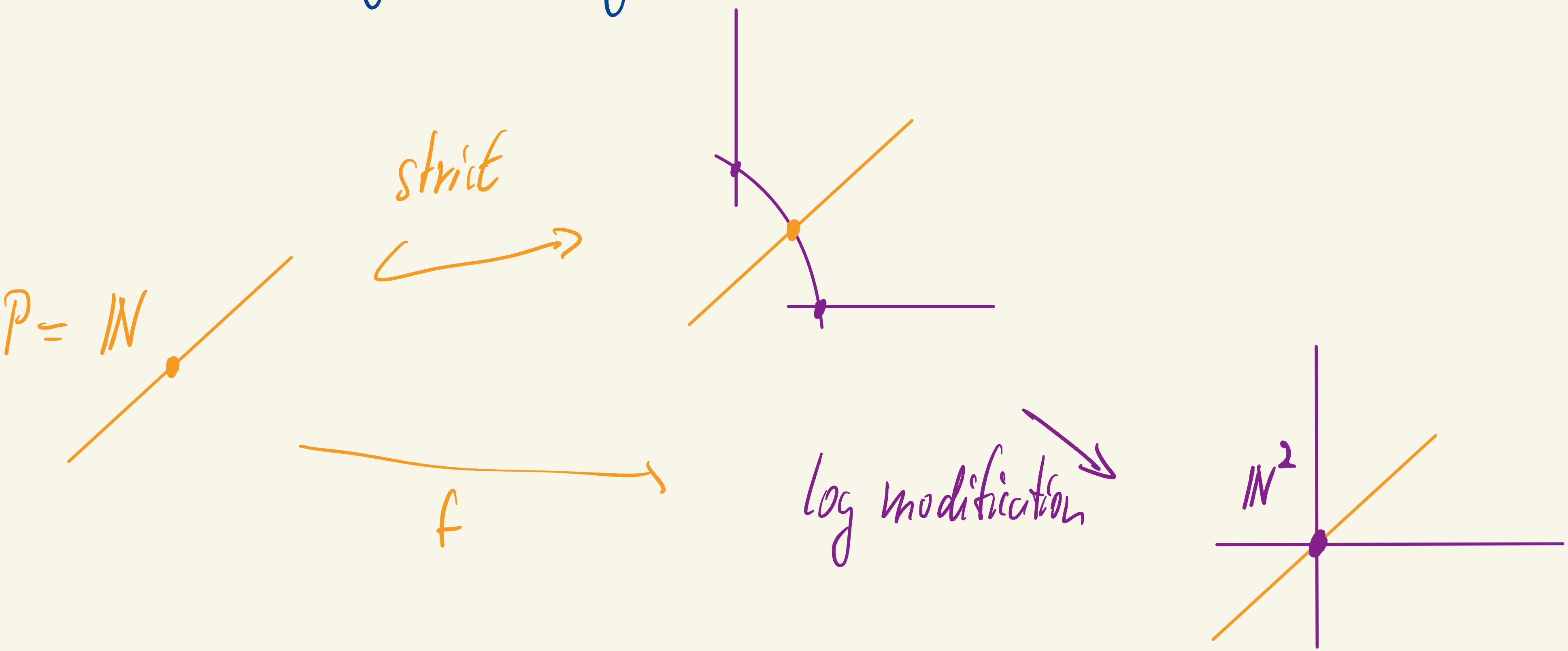
$P = \mathbb{N}^2$, $\mathbb{A}^1 \times \mathbb{G}_m^2 \to \mathbb{A}^2$

(ii) $Z = (\mathbb{A}^1, 0) \xrightarrow{f} X = (\mathbb{A}^2, V(zw))$

Z/k is log-flat:

$P = \mathbb{N}$, $\mathbb{A}^1 \times \mathbb{G}_m \to \mathbb{A}^1$

f is a closed log embedding:



Recall: closed subschemes

= equivalence class of closed embeddings

$$\begin{tikzcd} Z' \arrow[dr, hook, "i'"] \arrow[d, "\cong"] & \\ Z \arrow[r, hook, "i"'] & X \end{tikzcd}$$

Now: closed log subscheme = equivalence class of closed log embeddings, but equivalence induced by log modifications $\tilde{X} \to X$:

$$\begin{tikzcd} \tilde{Z} \arrow[r, "\tilde{i}"] \arrow[d] \arrow[dr, dashed, "\lrcorner" description] & \tilde{X} \arrow[d] \\ Z \arrow[r, "i"'] & X \end{tikzcd} \qquad (*)$$

Natural also from looking for proper monomorphisms:

Prop: $Z \xrightarrow{i} X$ is a proper monomorphism (in the category of fs-log schemes)

$\Leftrightarrow$ there exists a (fs-cartesian) diagram

$$\begin{tikzcd} \tilde{Z} \arrow[r, "\tilde{i}"] \arrow[d] & \tilde{X} \arrow[d] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$

strict closed embedding

log modif.

The stack of closed log subschemes

Thm The stack $\widetilde{\mathrm{LogHilb}}$ of closed log subschemes $= \varinjlim_{\Sigma} \mathcal{U}_{\Sigma}$ ← algebraic stack

Pf: Locally an open subscheme of $\mathrm{Hilb}_s(\tilde{X}_s) / \mathrm{Aut}(\tilde{X}_s / X_s)$

Pblm:
- $\widetilde{\mathrm{LogHilb}}$ is non-separated
- far from finite type even after fixing the Hilbert polynomial.

Common in log moduli pblms: Can always pull-back objects over log pts via

$$\mathrm{Spec}(Q \oplus \mathbb{N}^{\ell} \to k) \longrightarrow \mathrm{Spec}(Q \to k) \qquad [Q \hookrightarrow Q \oplus \mathbb{N}^{\ell},\ q \mapsto (q,0)]$$

In addition: $\varinjlim_{Y \text{ toric},\ \dim Y = n} Y$ not algebraic

III. Key question: local tropical moduli

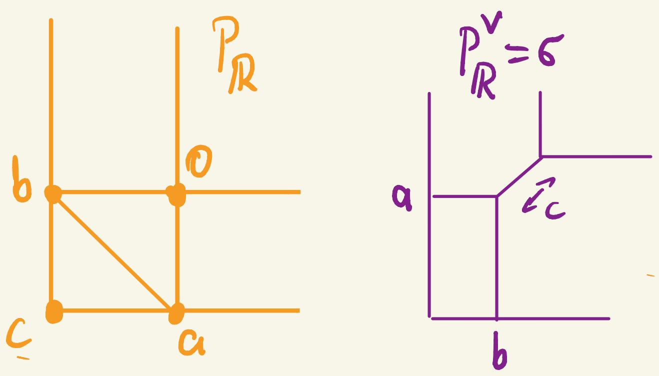
Expl: Tropical <u>hypersurfaces</u> in a cone $\Gamma \subseteq \sigma$

Γ = support of a balanced rational polyhedral complex

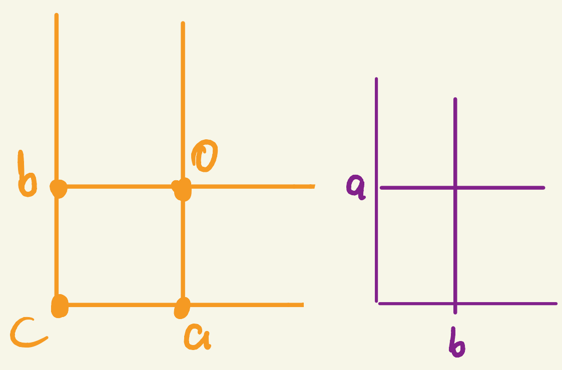
Task: Parametrize all Γ (of bounded degree) by a polyhedral complex

Solution: Secondary fan $f \in R[x_1,\dots,x_n] = R[P]$, $val: R \to \mathbb{R}_{\geq 0} \cup \{\infty\}$

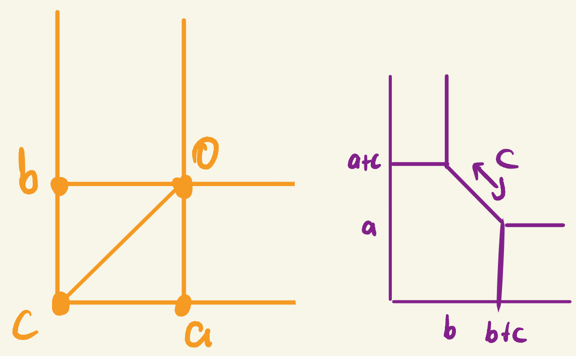
$n=2$, $f = xy + t^a x + t^b y + t^c$



$\boxed{a+b<c}$



$\boxed{a+b=c}$

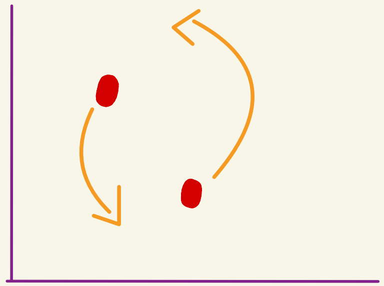


$\boxed{a+b>c}$

Q: How does this work in higher codimensions & in mixed dimensions?

Need to define a notion of type of tropical subspaces $\Gamma \leq G$ and define $\Gamma_1 + \Gamma_2$ for Γ_1, Γ_2 of the same type

Expl: a) two points in $\mathbb{R}^2_{\geq 0}$



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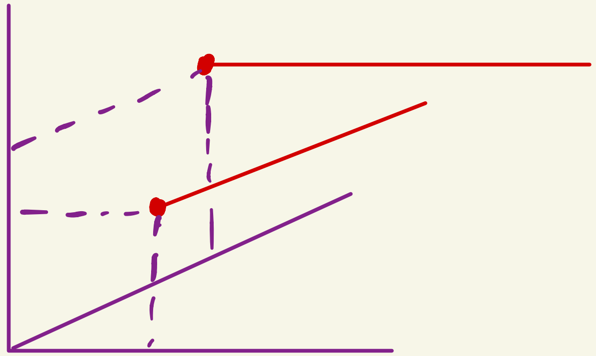
b) two skew lines in $\mathbb{R}^3_{\geq 0}$



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IV. Basic/minimal log structures

Expl 1: Stable curves $\rightsquigarrow$ M_g smooth, proper DM-stack $\rightsquigarrow$ Log structure $\mathcal{M}$ on M_g

$\cup$

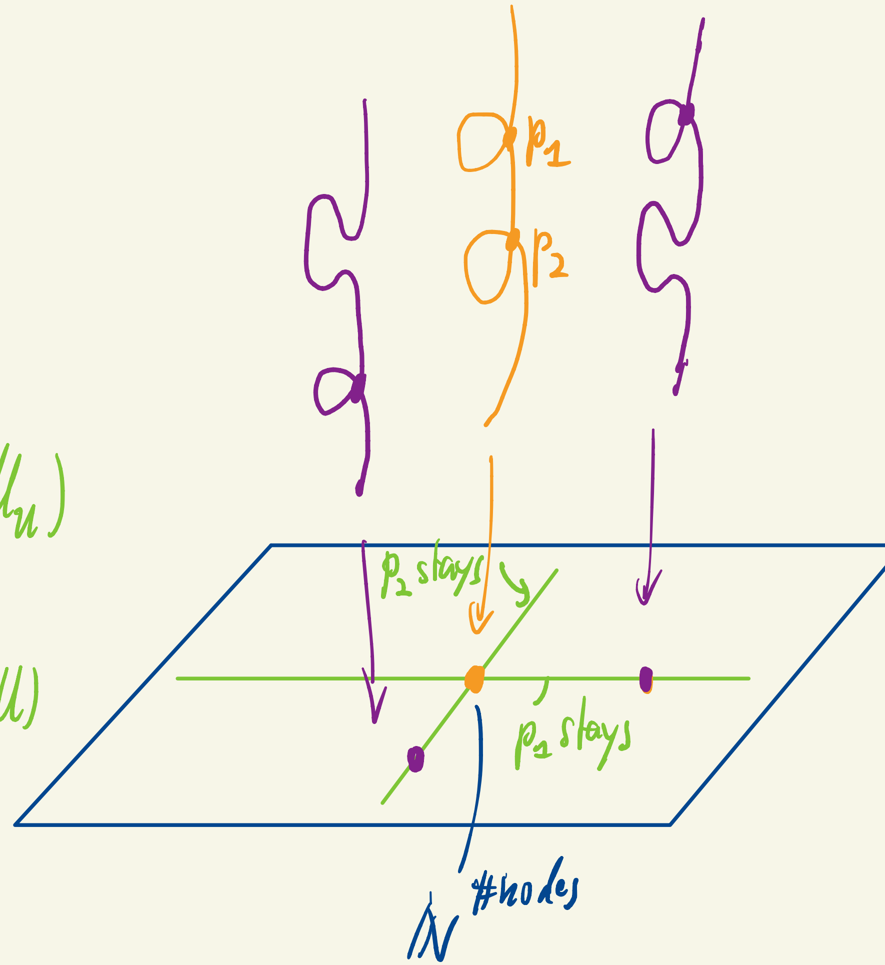
D_g nc divisor of nodal curves

stalk $\overline{\mathcal{M}}_C = \mathbb{N}^{\ell}$, ℓ = # nodes of C

Fact: Each (log smooth, integral, vertical) log str.

on $\begin{matrix} C \\ \downarrow \\ S \end{matrix}$ is unique log pull-back of the universal family

$$\begin{tikzcd} C \arrow[r] \arrow[d] & (U_g, \mathcal{M}_U) \arrow[d] \\ S \arrow[r] & (M_g, \mathcal{M}) \end{tikzcd}$$



Basic monoid: $Q = \mathbb{N}^{\ell}$ for an ℓ-nodal curve

Expl. 2: Stable log maps

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X & & \Gamma \arrow[r] \arrow[d] & \mathrm{Trop}(X) \\ \mathrm{Spec}(Q \to k) = S \arrow[rr, phantom, "\overset{\sim}{\text{trop}}"] & & & Q^\vee_{\mathbb{R}} = \mathrm{Hom}(Q \to \mathbb{R}_{\geq 0}) & \end{tikzcd}$$

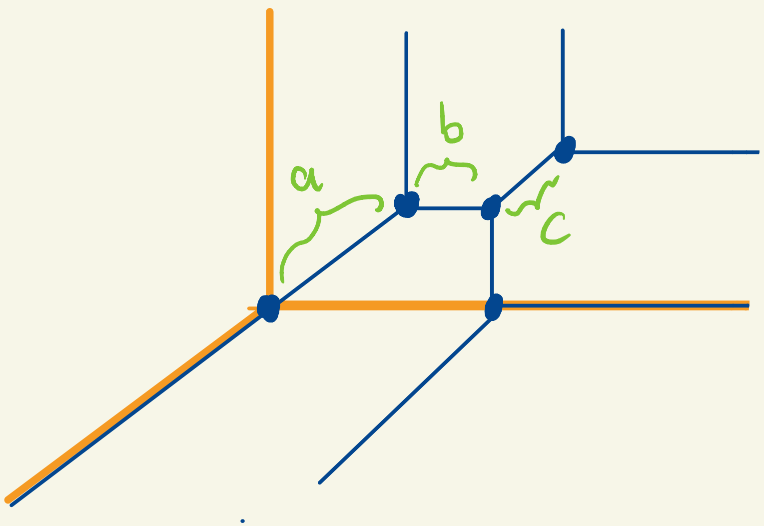
fibers = metric graphs

Basic monoid: Q = (dual of monoid of tropical stable maps of the same type)$_{\mathbb{Z}}$

Use functorial tropicalization of log schemes: $\mathrm{Trop}(X) = \varinjlim_{x \in X} (\overline{M}_{X,x})^\vee_{\mathbb{R}}$

$[P^\vee_{\mathbb{R}} = \mathrm{Hom}_{\mathrm{Mon}}(P, \mathbb{R}_{\geq 0})]$

Expl: A conic in $X = \mathbb{P}^2$



$\rightsquigarrow \quad Q^\vee_{\mathbb{R}} = \mathbb{R}^3_{\geq 0} \ni (a,b,c), \quad Q = \mathbb{N}^3$

V. Basic monoids for Log Hilb and hyperplane arrangements

Closed log embedding

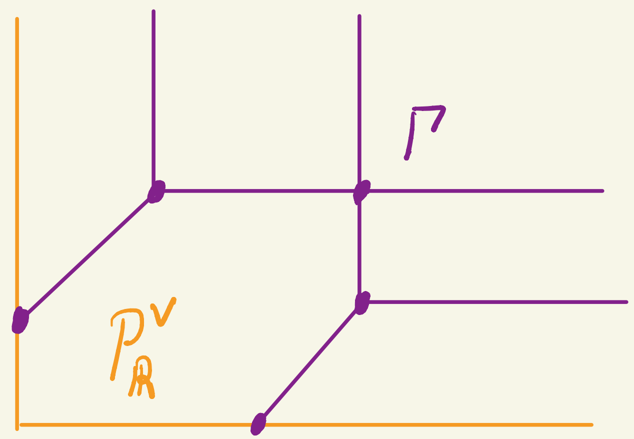
$$\begin{tikzcd} Z \arrow[r] & X_s \arrow[d] \\ & S = \mathrm{Spec}(Q \to k) \end{tikzcd} \rightsquigarrow$$ family of trop. subspaces $\{\Gamma_s\}$ [one for each assoc. prime]

$$\begin{tikzcd} \Gamma_s \arrow[r, hook] \arrow[d] & \Gamma = \mathrm{Trop}(Z) \arrow[r, "2"] \arrow[d] & \mathrm{Trop}(X) \\ \{s\} \arrow[r, hook] & Q^\vee_{\mathbb{R}} & \end{tikzcd}$$ cone

Pblm: Polyhedral decomposition of Γ_s changes under equivalence of log embeddings.

Locally in X: $\Gamma \subseteq P^\vee_{\mathbb{R}}$ support of balanced polyhedral complex $\mathcal{P}$

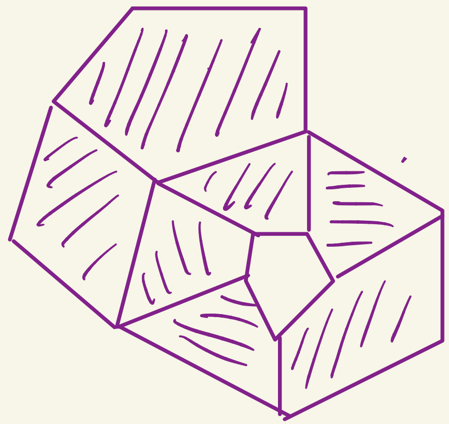
$\mathcal{P} = \bar{\mathcal{M}}_{X,x}$ [depends on choice of $\tilde{X}_s$!]



Flats

$F \subseteq \Gamma$ closure of connected component of Γ_{reg}

flats may not be convex



Hyperplanes associated to flat F:

$$H_{F,\tau} = F + TF + T\tau\,, \quad \tau \subseteq P_{\mathbb{R}}^{\vee} \text{ face s.th. } \dim H_{F,\tau} = \operatorname{rk} P - 1.$$

$P_{\mathbb{R}}^{\vee} = \sigma$ for one stratum of X

Hyperplane arrangement for Γ:

$$\mathcal{H}_{\Gamma} = \{ H_{F,\tau} \mid F, \tau \}$$

[in $P_{\mathbb{R}}^{\vee}$ or in $P_{\mathbb{R}}^{*}$]

$\mathcal{H}_{\Gamma}$ defines a polyhedral decomposition $\mathcal{P}_{\Gamma}$ of $P_{\mathbb{R}}^{\vee}$

Lemma: Each flat $F \subseteq \Gamma$ is a union of cells of $\mathcal{P}_{\Gamma}$. [uses that Γ is balanced]

Type of Trop(Z):

At each $x \in X$, $P = \mathfrak{M}_{X,x}$:

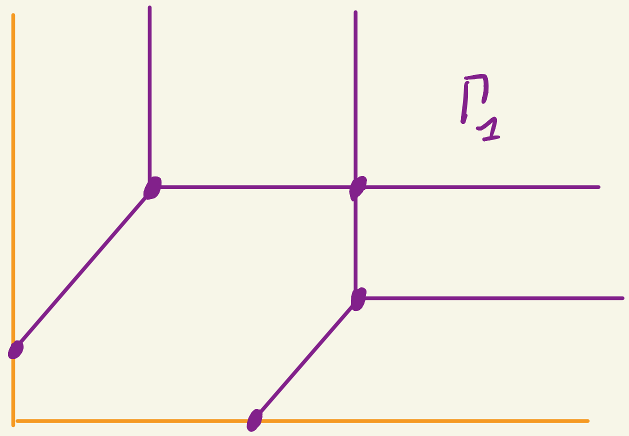
- hyperplane arrangement $\mathcal{H}_x = \bigcup_{\Gamma = \mathrm{Trop}(Z')} \mathcal{H}_\Gamma$ in $\mathbb{P}^\vee_{\mathbb{R}}$, $Z' \subseteq Z$ embedded cmp.

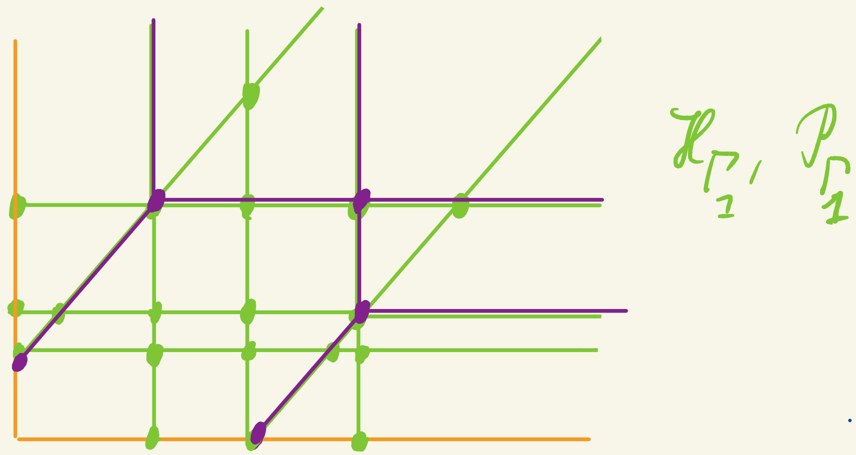
- associated polyhedral decomposition $\mathcal{P}_x$

- type of $\mathcal{P}_x$: category of cells & star at each vertex (a fan in $P^*_{\mathbb{R}}$).

- subcategory of cells of $\mathcal{P}_x$ covering Γ.

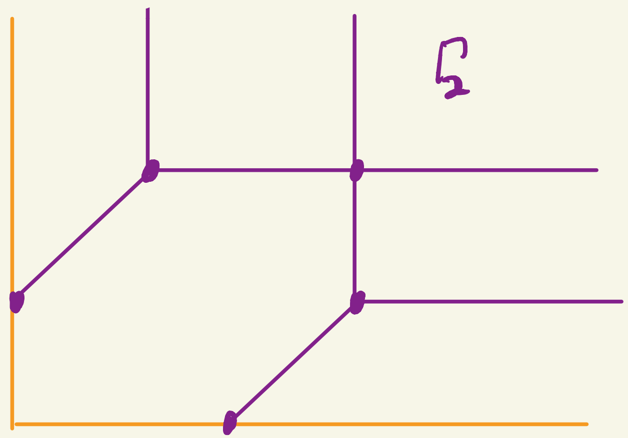
Example Tropical curves of different <u>embedded</u> type

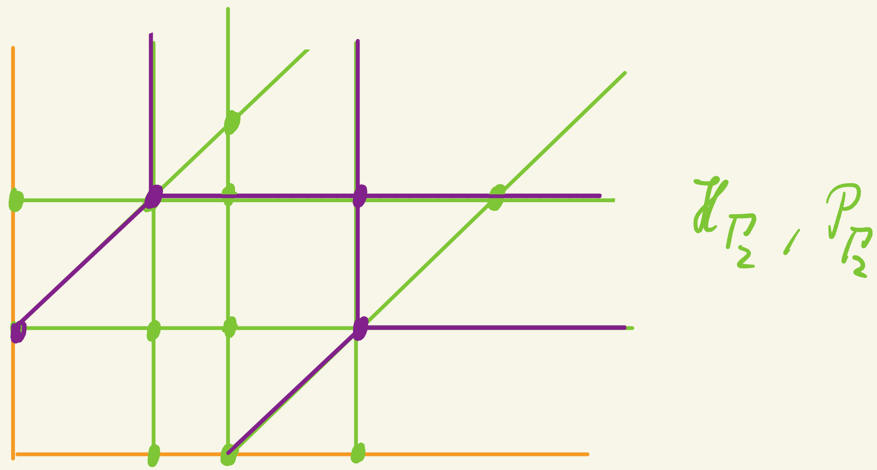




$\updownarrow$ same type in log GW

$\updownarrow$ different type in Log Hilb





<u>Key Lemma</u>

- tropical subspaces of the same type can be added
  $\to$ rational polyhedral cone
- addition is compatible with generization maps in X:
  $x \in \overline{y} \Rightarrow P^\vee_{y,\mathbb{R}} \hookrightarrow P^\vee_{x,\mathbb{R}}$ face

  $\to$ basic monoid Q_{bas}
- basicness is an open property in $\widetilde{\mathrm{LogHilb}} \rightsquigarrow \mathrm{LogHilb} \underset{\text{open}}{\subseteq} \widetilde{\mathrm{LogHilb}}$

<u>Suggests:</u> Tropical Hilbert scheme TropHilb based on hyperplane arrangements

$\rightsquigarrow$ $\mathrm{Trop}: \mathrm{LogHilb} \longrightarrow \mathrm{TropHilb}$