K-moduli space of del pezzo surface pairs

Joint work with Long Pan and Haoyu Wu

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Background

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Definition

K3 surface is a smooth projective surface with $H^1(S, \mathcal{O}_S) = 0$ and $K_S \sim \mathcal{O}_S$.

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- Anti-canonical sections of Fano 3-fold. For example, X is a prime Fano 3-fold with $-K_X \sim rH$ and $S \in |-K_X|$ general, then $(S, H|_S)$ is a polarised K3 surface of degree $(H|_S)^2 = 2g - 2$.

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- Double cover of del pezzo surface. Let X be a del pezzo surface of degree $d = (-K_X)^2$ and

$$\varphi : S \to X$$

  double cover branched along a curve $C \in |-2K_X|$. Then $(S, \tau : S \to S)$ is a K3 surface with anti-symplectic involution.

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$$M \hookrightarrow \mathcal{F}_\Lambda := \Gamma_\Lambda \setminus \mathcal{D}_\Lambda$$

  for lattice Λ of signature $(2, n)$ and Γ_Λ monodromy group. Then M has Baily-Borel compactification $\mathcal{F}_\Lambda^*$. For example, $S \in |-K_X|$ and then $\Lambda_g \cong E_8^2 \bigoplus U^2 \bigoplus \langle 2-2g \rangle$ and $\mathcal{F}_g^*$.

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- K-moduli side: $P^K_c = \{(X, cS) \mid K\text{-polystable pairs}\}$.

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- Xu in his survey article also asks how to compare the K-moduli of prime Fano 3-folds and compactifications of polarised K3 surfaces of degree $2g-2$.
- A general expectation is that K-moduli wall-crossing will give an explicit resolution of the birational period map

$$p : \overline{M}^{GIT} \dashrightarrow \mathcal{F}_{\Lambda}^{*}$$

.

Known exmples

- Ascher-DeVleming-Liu 2019:

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$$|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4)| /\!\!/ PGL(2)\times PGL(2) \dashrightarrow \mathcal{F}^*$$

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In this talk, we focus on another example: Double cover $X \to \mathbb{F}_1 \cong Bl_p\mathbb{P}^2$.

K-stability

Definition

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$$\beta_{(X,D)}(E) := A_{(X,D)}(E) - S_{(X,D)}(E) \geq 0$$

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If the pair (X, D) is of complexity one, then

Theorem (Zhuang, Ilten-Suss, ACC+)

Let (X, D) be a 2-dimensional log Fano pair with an effective $\mathbb{G}_m$-action λ. Then (X, D) is K-polystable if and only if the followings hold:

1. $\beta_{(X,D)}(F) > 0$ *for all vertical λ-invariant prime divisors F on X;*
2. $\beta_{(X,D)}(F) = 0$ *for all horizontal λ-invariant prime divisors F on X;*
3. $\beta_{(X,D)}(v) = 0$ *for the valuation v induced by the 1-PS λ.*

K-moduli

By many people's work, the moduli stack of K-semistable log Fano pairs (X, cD) has good moduli space

$$P_c^K = \{(X, cD) \mid K\text{-polystable pairs}\}$$

where $D \sim -mK_X$ and X is $\mathbb{Q}$-Fano. In this talk, we consider $m = 2$.

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Theorem (Ascher-DeVleming-Liu- 2019)

There are finitely many rational numbers (i.e., walls) $0 < w_1 < \cdots < w_m < \frac{1}{2}$ such that

$$\overline{P}_c^K \cong \overline{P}_{c'}^K \ \text{ for any } w_i < c, c' < w_{i+1} \text{ and } \text{ any } 1 \leq i \leq m-1.$$

Denote $\overline{P}_{(w_i,w_{i+1})}^K := \overline{P}_c^K$ for some $c \in (w_i, w_{i+1})$, then at each wall w_i there is a flip (or divisorial contraction)

$$\overline{P}_{(w_{i-1},w_i)}^K \longrightarrow \overline{P}_{w_i}^K \longleftarrow \overline{P}_{(w_i,w_{i+1})}^K$$

which fits into a local VGIT.

Locally symmetric varieties $\mathcal{F}$ associated to degree 8 log Fano pairs

Generically, $X \to \mathbb{F}_1 \cong Bl_p\mathbb{P}^2$ has following Neron-Severi group

$$NS(X) = \left(\begin{array}{c|cc} & L & E \\ \hline L & 2 & 0 \\ E & 0 & -2 \end{array}\right)$$

$\Lambda := U^2 \oplus E_7 \oplus E_8 \oplus A_1 \cong (NS(X) \hookrightarrow H^2(X,\mathbb{Z}))^{\perp}$. Define

$$\mathcal{D} := \{z \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid z^2 = 0, z.\overline{z} > 0\}^{+}, \quad \Gamma := O^{+}(\Lambda)$$

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- $\mathcal{F}$ has Baily-Borel compactification $\mathcal{F}^*$

$$\mathcal{F}^* - \mathcal{F} = \bigcup B_I$$

Moduli of del pezzo pair of degree 8

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$$D = \{z^4 f_2(x,y) + z^3 f_3(x,y) + \cdots + f_6(x,y) = 0\}.$$

Assume $f_2(x,y)$ has rank 2, then curve D has the form

$$az^4 xy + z^3 \widetilde{f_3}(x,y) + z^2 f_4(x,y) + zf_5(x,y) + f_6(x,y) = 0$$

Let $\mathbb{P}V$ be the parameter space of such D and then GIT space $\mathbb{P}V /\!\!/ T$ provides a partial compactification for P.

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- P has (at least partially) a series of compactifications P_c^K via viewed as a log Fano pair $(\mathbb{F}_1, cC)$.

Two divisors $\mathcal{F}_\Lambda$

- Hyperelliptic divisor H_h: a general element in H_h is X as a double of $Bl_p\mathbb{P}^2$ branched along a general curve $C \in |-2K_{Bl_p\mathbb{P}^2}|$ tangent the (-1)-curve E.

$$NS(X) = \left(\begin{array}{c|ccc} & L & E_1 & E_2 \\ \hline L & 2 & 0 & 0 \\ E_1 & 0 & -2 & 1 \\ E_2 & 0 & 1 & -2 \end{array}\right)$$

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- Unigonal divisor H_u: a general element in H_u is X as a double of minimal resolution $Bl_p\mathbb{P}(1,1,4)$.

$$NS(X) = \left(\begin{array}{c|ccc} & E' & F' & H'_y \\ \hline E' & -2 & 0 & 2 \\ F' & 0 & -2 & 1 \\ H'_y & 2 & 1 & -2 \end{array}\right)$$

Main results 1

Theorem (Pan-Si-Wu,2023)

1. *The walls for K-moduli space* P_c^K *are*

$$W_h = \{ \frac{1}{14}, \frac{5}{58}, \frac{1}{10}, \frac{7}{62}, \frac{1}{8}, \frac{5}{34}, \frac{1}{6}, \frac{7}{38}, \frac{1}{5}, \frac{5}{22}, \frac{2}{7} \}$$
$$W_u = \{ \frac{29}{106}, \frac{31}{110}, \frac{2}{7}, \frac{35}{118} \}$$

2. *If* $c \in (0, \frac{1}{14})$, P_c^K *is empty. If* $c \in [\frac{1}{14}, \frac{5}{58})$,

$$\mathrm{P}_c^K \cong \mathbb{P}V /\!\!/ T$$

Main results 1, continued

Theorem (Pan-Si-Wu,2023)

1. *There are two divisorial contraction morphism* $\mathrm{P}^K_{w+\epsilon} \to \mathrm{P}^K_w$ *at wall* $w = \frac{5}{58}$ *and* $w = \frac{29}{106}$. *The exceptional divisors* E^+_w *is birational to hyperelliptic divisor* H_h*(resp. unigonal divisor* H_u *).*

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2. *There is arithmetic stratification*

$$\cdots \subset NL_{h,A_3} \subset NL_{h,A_2} \subset H_h$$

of Noether-Lefschetz locus on H_h*, which are proper transform of* E^+_w *for* $w \in W_h$*. Similar arithmetic stratification on* H_u *and the strata are birational to* E^+_w *for* $w \in W_u$*.*

Table for K-wall

wall	curve B on $\mathbb{P}^2$	weight	curve singularity at p
$\frac{1}{14}$	$x^4zy=0$	(1,0,0)	A_1
$\frac{5}{58}$	$x^4z^2+x^3y^3=0$	(0,2,3)	A_2
$\frac{1}{10}$	$x^4z^2+x^3zy^2+a\cdot x^2y^4=0$	(0,1,2)	A_3
$\frac{7}{62}$	$x^4z^2+xy^5=0$	(0,2,5)	A_4
$\frac{1}{8}$	$x^4z^2+x^2zy^3+a\cdot y^6=0,$	(0,1,3)	A_5 tangent to L_z
	$x^3f_3(z,y)=0$	(0,1,1)	D_4
$\frac{5}{34}$	$x^4z^2+xzy^4=0$	(0,1,4)	A_7 with a line
	$x^3z^2y+x^2y^4=0$	(0,2,3)	D_5
$\frac{1}{6}$	$x^4z^2+zy^5=0$	(0,1,5)	A_9 with a line
	$x^3z^2y+x^2zy^3+a\cdot xy^5=0$	(0,1,2)	D_6

Table: K-moduli walls from Gorenstein del Pezzo $\mathbb{F}_1=Bl_{[1,0,0]}\mathbb{P}^2$

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$\frac{1}{5}$	$x^3z^2y + xzy^4 = 0$	(0,1,3)	D_8 with L_z
$\frac{5}{22}$	$x^3z^2y + zy^5 = 0$	(0,1,4)	D_{10} with L_z
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wall	curve B on $\mathbb{P}(1,1,4)$	weight	(a,b,m)
$\frac{29}{106}$	$z^3 + z^2x^4 = 0$	(1,0,4)	$(0,1,0)$
$\frac{31}{110}$	$z^3 + zyx^7 = 0$	(2,0,7)	$(1,1,1)$
$\frac{2}{7}$	$z^3 + y^2x^{10} = 0$	(3,0,10)	$(2,1,2)$
$\frac{35}{118}$	$z^3 + zy^2x^6 + y^3x^9 = 0$	(1,0,3)	(1,0,1)

Table: K-moduli walls from index 2 del Pezzo $Bl_{[1,0,0]}\mathbb{P}(1,1,4)$

Main results 2

Define the Hasset-Keel-Looijenga (HKL) model for $\mathcal{F}^*$

$$\mathcal{F}(s) := \mathrm{Proj}(\bigoplus_m H^0(\mathcal{F}^*, m(\lambda + sH_h + 25sH_u))$$

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Theorem (Pan-Si-Wu,2023)

There is natural isomorphism $P_c^K \cong \mathcal{F}(s)$induced by the period map under the transformation

$$s = s(c) = \frac{1-2c}{56c-4}$$

where $\frac{1}{14} < c < \frac{1}{2}$. In particular, P_c^K will interpolates the GIT space $\overline{P}^{GIT}$ and Baily-Borel compactification $\mathcal{F}^*$. In particular, walls are $w = \frac{1}{n}$ and

$$n \in \{1, 2, 3, 4, 6, 8, 10, 12, 16, 25, 27, 28, 31\}$$

Sketch of proof of main results 1

- Step1: To determine K-semistable degeneration. (X, cD) has T-singularities at worst.

  $$\frac{32}{9}(1-2c)^2 \leq \widehat{vol}(X, cD; x)$$

  Combining index 1 covering trick, $ind(K_X, x) \leq 3$.
  By Nakayama, Fujita-Yasutake's classification results of index ≤ 3 del pezzo surface, we rule out index 3 case by showing they are K-unstable. index ≤ 2 case:

  $$Bl_p\mathbb{P}^2, \quad Bl_p\mathbb{P}(1,1,4).$$

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$$Bl_p\mathbb{P}^2, \quad Bl_p\mathbb{P}(1,1,4).$$

- Step2: Local VGIT structure of K-moduli implies if $(Bl_p\mathbb{P}^2, C)$ or $(Bl_p\mathbb{P}(1,1,4), C)$ in the center, then it admits 1-PS λ and thus $Fut(\lambda) = \beta(F)$ where F is an exceptional divisor of certain weighted blowup determined by λ.

- Step2 continued: For example, for some λ,

$$A_{(X,cC)}(F) = a + b - mc, \quad S_{(X,cC)}(F) = \frac{106b + 83a}{48}(1 - 2c)$$

Then $A_{(X,cC)}(F) = S_{(X,cC)}$ will give us all potential walls. Then using equivariant K-stability criterion to determine which potential wall is a real wall.

- Step3: To determine the 1st walls and then keep track of wall crossing at all centers for each walls. Following the arguments of Liu-Xu, show for c small and any K-degeneration (X_0, cC_0) of $(Bl_p\mathbb{P}^2, cC)$, X_0 is still $Bl_p\mathbb{P}^2$, then can show

$$\mathrm{P}_c^K \cong \mathbb{P}V /\!\!/ T.$$

Then explicit wall-crossing are followed by analysis of local VGIT at each wall $w \in W_u \cup W_h$.

Sketch of proof of main results 2

- step 1: By ampleness of CM line bundle and birational contraction map

$$\overline{P}^K_{\frac{1}{2}-\epsilon} \dashrightarrow \overline{P}^K_c,$$

Then $\overline{P}^K_c \cong \mathrm{Proj}(R(\overline{P}^K_{\frac{1}{2}-\epsilon}, \lambda_{\frac{1}{2}-\epsilon,c})$ where

$$\lambda_{\frac{1}{2}-\epsilon,c} := \pi_*(-K_{\mathfrak{X}} + c\mathcal{C})^3$$

where $(\mathfrak{X}, \mathcal{C})$ is universal family of pairs on $\overline{P}^K_{\frac{1}{2}-\epsilon}$.
Then enough to show $(p^{-1})^*\lambda_{\frac{1}{2}-\epsilon,c}$ on $\mathcal{F}^*$ is proportional to

$$\lambda + \frac{1-2c}{56c-4}(H_h + 25H_u).$$

where $p : \overline{P}^K_{\frac{1}{2}-\epsilon} \dashrightarrow \mathcal{F}^*$ birational period map.

- step 2: Applying interpolation formula of CM line bundles

$$(1-2c)^{-2}\cdot\lambda_{\frac{1}{2}-\epsilon,c} = (1-2c)\cdot\lambda_{\frac{1}{2}-\epsilon,0} + 48c\cdot\lambda_{\frac{1}{2}-\epsilon,Hdg}.$$

$p^{-1\ *}\lambda_{\frac{1}{2}-\epsilon,Hdg} = \lambda$ and it remains to determine

$$p^{-1\ *}\lambda_{\frac{1}{2}-\epsilon,0} = a_h H_h + a_u H_u + a_\lambda\lambda,\ \ a_u, a_h, a_\lambda \in \mathbb{Q}$$

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- step 3: The coefficient $a_u,a_h,a_\lambda\in\mathbb{Q}$ are determined by walls

$$\frac{1}{14},\frac{5}{58},\frac{29}{106}.$$

Denote the $\mathbb{Q}$-line bundle

$$\Delta(c):=48c\lambda+(1-2c)\cdot(a_hH_h+a_uH_u+a_\lambda\lambda)$$

Then any multiple of $\Delta(c)$ has no global sections at wall $\frac{1}{14}$. This shows $a_\lambda=-4$.

- step 3, continued: at wall $\frac{5}{58}$ where hyperellitpic divisor appears, $\Delta(\frac{5}{58})|_{H_h} = 0$.
  This shows $a_h = 1$. Similar arguments will show $a_u = 25$.
  A key input is that the computation

$$(\lambda + H_u)|_{H_u} = 0, \quad (\lambda + H_h)|_{H_h} = 0$$

  via Borcherds' work automorphic forms on locally symmetric varieity $\mathcal{F}$, which gives the relation of Heegner divisors on $\mathcal{F}$. In our case ,

$$76\lambda = H_n + 2H_h + 57H_u.$$

Some remarks:

- For higher dimensional log Fano pairs, to find walls of their K-moduli seems much harder than dimension 2. The arithmetic stratifications should be powerful to predict walls for K-moduli of log Fanos related to K3 surfaces (even irreducible holomorphic symplectic varieties).

Some remarks:

- For higher dimensional log Fano pairs, to find walls of their K-moduli seems much harder than dimension 2. The arithmetic stratifications should be powerful to predict walls for K-moduli of log Fanos related to K3 surfaces (even irreducible holomorphic symplectic varieties).
- It should be interesting to look at the behavior of $c > \frac{1}{2}$ and $c = \frac{1}{2}$. For $c > \frac{1}{2}$, by Alexeev-Engel-Han's work, the KSBA moduli space compactifying pairs $(Bl_p\mathbb{P}^2, cC)$ and their slc degeneration has a natural normalization— Toroidal compactification of $\mathcal{F}$.
  For $c = \frac{1}{2}$, it is expected to have a moduli theory for log CY to connect wall crossing from K-moduli to KSBA moduli.

Thank you for your attention !