Polyhedral Divisors and Orbit Decompositions of Normal Affine T-varieties

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POLYHEDRAL DIVISORS AND ORBIT DECOMPOSITIONS OF NORMAL AFFINE T-VARIETIES

JOINT WORK WITH KLAUS ALTMANN
BASED ON ARXIV: MATH/0306285

GOAL ORBIT DECOMPOSITION OF NORMAL AFFINE T-VARIETIES

$X \circlearrowleft T$ EFFECTIVE

TOOL COMPLETE DESCRIPTION OF SUCH $X^n \circlearrowleft T^k$ IN TERMS OF SO-CALLED "PROPER POLYHEDRAL DIVISORS" ON A NORMAL SEMIPROJECTIVE VAR. Y^{n-k}.

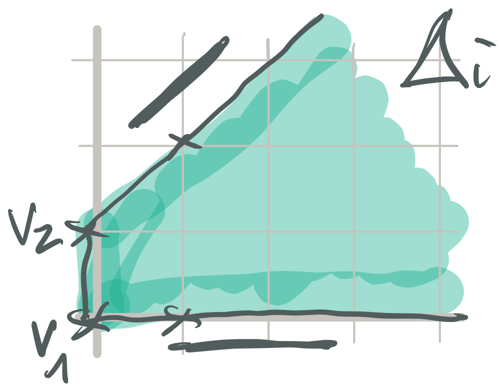
SETUP [CONSTRUCTION OF X]

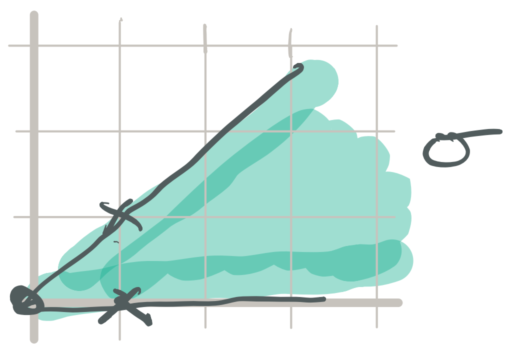
- Y NORMAL SEMIPROJ.
- $\mathcal{D}$ POLYHEDRAL DIVISOR ON Y,

$$\mathcal{D} = \sum_i \underset{\subseteq N_{\mathbb{Q}}}{\Delta_i} \otimes \underset{\subseteq Y \text{ PRIME DIV.}}{\underbrace{D_i}}$$

POLYHEDRA WITH COMMON

$$\mathrm{TAIL}(\Delta_i) = \underset{\substack{\cap\!\!| \\ N_{\mathbb{Q}}}}{\sigma} \leftarrow \text{POINTED CONE}$$





REMARK

$$\underset{\text{POLYHEDRON}}{\Delta} = \overset{\text{POLYTOPE}}{\Pi} + \sigma \leftarrow \text{UNIQUE TAILCONE}$$

HERE: $\Pi = \mathrm{CONV}(v_1, v_2)$

- EVALUATION MAP

$$\sigma^\vee \longrightarrow \mathrm{Div}_{\mathbb{Q}}(Y)$$
$$u \longmapsto \mathcal{D}(u) := \sum_i \min_{v \in \Delta_i} \langle u, v \rangle D_i$$

- $\mathcal{D}$ PROPER POLYHEDRAL DIVISOR

:$\Leftrightarrow$ 1) $\mathcal{D}$ POLYH. DIVISOR

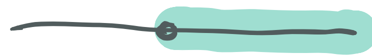
2) $\mathcal{D}(u) \in \mathrm{CaDiv}_{\mathbb{Q}}(Y)$ $\forall u \in \sigma^\vee$

3) $\mathcal{D}(u)$ SEMIAMPLE $\forall u \in \sigma^\vee$

4) $\mathcal{D}(u)$ BIG $\forall u \in \mathrm{RELINT}(\sigma^\vee)$

EXAMPLE

$=\sigma^\vee$

$Y = \mathbb{P}^1$, $N = \mathbb{Z}$, $\sigma = [0,\infty) =$ 

$$\mathcal{D} = \underbrace{[-\tfrac{1}{3},\infty)}_{-\frac{1}{3}+\sigma} \otimes \{0\} + \underbrace{[\tfrac{1}{2},\infty)}_{\frac{1}{2}+\sigma} \otimes \{\infty\}$$

$$\mathcal{D}(\underset{\substack{\vee\!\mid \\ 0}}{u}) = -\frac{u}{3}\{0\} + \frac{u}{2}\{\infty\}$$

3)+4): $\underbrace{\sum \mathrm{DEG}(\mathcal{D}_i)\,\Delta_i}_{= +\frac{1}{6}+\sigma \ \checkmark} \subseteq \mathrm{RELINT}(\sigma)$

$\therefore$ $\mathcal{D}$ PP-DIVISOR ON Y WITH TAIL CONE σ

- $A = \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, \mathcal{O}(\mathcal{D}(u)))$ FIN. GEN M-GRADED $\mathbb{C}$-ALGEBRA

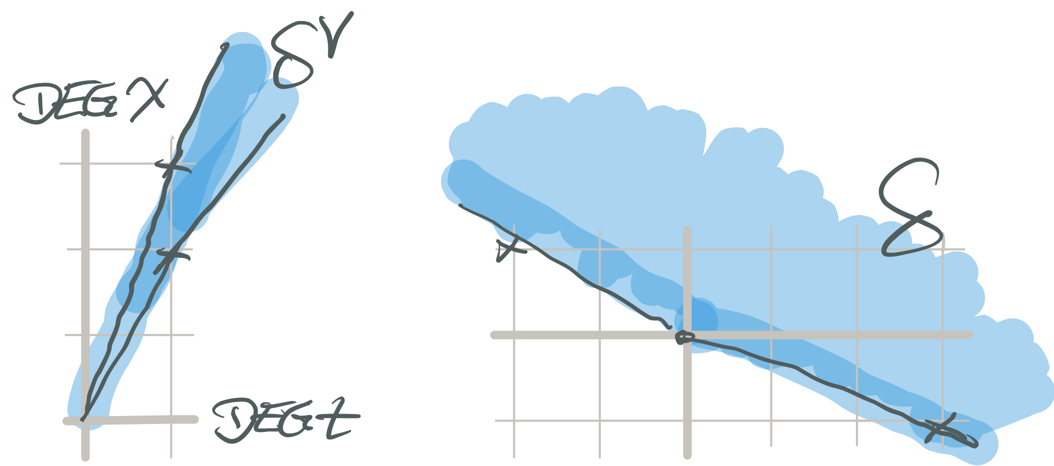
- $X = \mathrm{SPEC}(A)$ NORMAL AFFINE T-VARIETY
  $T = \mathrm{SPEC}(\mathbb{C}[M])$

WHERE THE ACTION IS GIVEN BY THE M-GRADING OF A.

$$\dim(X) = \dim(Y) + \dim(T)$$

EXAMPLE

$$A = \bigoplus_{u \geq 0} \Gamma(\mathbb{P}^1, \mathcal{O}(\mathcal{D}(u))) \chi^u = \mathbb{C}[t\chi^2, t\chi^3]$$



$$X = \mathrm{SPEC}(A) = \mathbb{C}^2 = TV(\delta)$$

$$T \times X \longrightarrow X, \quad (t, (x, y)) \longmapsto (t^2 x, t^3 y)$$

REMARK EVERY NORMAL ~~AFFINE~~ T-VARIETY WITH AN EFFECTIVE ACTION ARISES FROM A PP-DIV. ON A NORMAL SEMIPROJ. VAR.

INTERMEDIATE STEP

- $\mathcal{A} = \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}(\mathcal{D}(u))$ $\mathcal{O}_Y$-ALGEBRA
- $\tilde{X} = \mathrm{SPEC}_Y(\mathcal{A})$ NORMAL VAR. WITH EFFECTIVE T-A.

$$\begin{tikzcd}
\tilde{X} \arrow[r, "r"] \arrow[d, "\pi"', "\text{GOOD QUOTIENT}"] & X \\
Y &
\end{tikzcd}$$

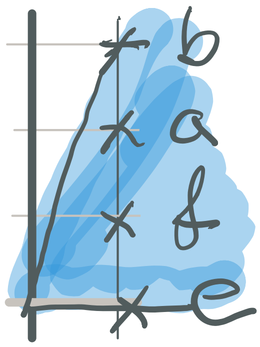
PROPER BIRATIONAL T-EQUIVARIANT CONTRACTION

EXAMPLE $\mathcal{A} = \bigoplus_{u \geq 0} \mathcal{O}_{\mathbb{P}^1}(\mathcal{D}(u))\, \chi^u$

COMPUTED LOCALLY ON $U_1 := \mathbb{P}^1 \setminus \{\infty\}$:

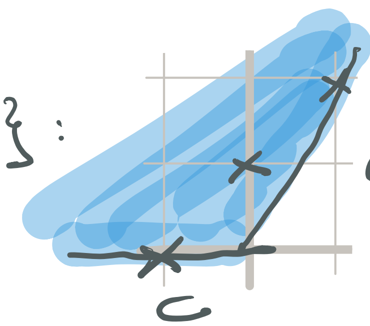
$$\mathcal{O}_{\mathbb{P}^1}(\mathcal{D}(u))(U_1) = \frac{1}{t^{\lfloor -\frac{u}{3} \rfloor}} \mathbb{C}[t]$$

$$\Rightarrow t^g \chi^u \in \frac{1}{t^{\lfloor -\frac{u}{3} \rfloor}} \mathbb{C}[t]\, \chi^u \iff 3g - u \geq 0$$

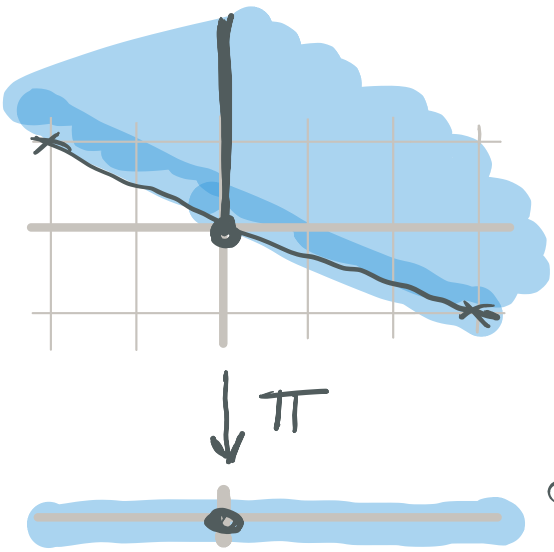


$$\mathbb{C}[b,a,f,e] \Big/ \begin{matrix} eb = af \\ f^2 = ea \\ a^2 = fb \end{matrix}$$

AND ON $U_2 := \mathbb{P}^1 \setminus \{0\}$:



$$\mathbb{C}[a,d,c] \big/ ac = d^2$$



$\mathbb{P}^1$

$\tilde{X}$ SINGULAR

T-ORBITS OF $\tilde{X}$

$$\left\{ \begin{array}{l} (y,F) \mid y \in Y \\ F \leq \underbrace{\Delta_y} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{T-ORBITS} \\ \text{IN } \tilde{X} \end{array} \right\}$$

$= \sum_{y \in D_i} \Delta_i$ FIBER POLYHEDRON

DIM = CODIM F

$(y, F) \longmapsto ORB_{\tilde{X}}(y, F) \subseteq \pi^{-1}(y)$

UNIQUE PAR. BY F

T-ORBITS OF X

$$\left\{ \begin{array}{l} (y,F) \mid y \in \mathbb{P}^1 \\ F \leq \Delta_y \end{array} \right\} \twoheadrightarrow \left\{ \begin{array}{l} \text{T-ORBITS} \\ \text{OF X} \end{array} \right\}$$

$(y, F) \longmapsto \underbrace{ORB_X(y, F)}$

$= r(ORB_{\tilde{X}}(y, F))$

EXAMPLE

T-ORBITS OF $\tilde{X}$

$y \in \mathbb{P}^1$	Δ_y	$ORB_{\tilde{X}}(y, F \preceq \Delta_y) \subseteq \pi^{-1}(y)$
0	$-\frac{1}{3} + \sigma$	$ORB_{\tilde{X}}(0, -\frac{1}{3})$ OF DIMENSION 1 $ORB_{\tilde{X}}(0, \Delta_0)$ — " — 0
∞	$\frac{1}{2} + \sigma$	$ORB_{\tilde{X}}(\infty, \frac{1}{2})$ $ORB_{\tilde{X}}(\infty, \Delta_\infty)$
$y \neq 0, \infty$	σ	$ORB_{\tilde{X}}(y, 0)$ $ORB_{\tilde{X}}(y, \Delta_y)$

T-ORBITS OF X

$y \in \mathbb{P}^1$	Δ_y	$ORB_X(y, F \preceq \Delta_y)$
0	Δ_0	$ORB_X(0, -\frac{1}{3}) = T \cdot (a, 0)$ $= \{(t^2 a, 0) \mid t \in \mathbb{C}^*\} \cong \mathbb{C}^*$
∞	Δ_∞	$ORB_X(\infty, \frac{1}{2}) = T \cdot (0, b)$ $= \{(0, t^3 b) \mid t \in \mathbb{C}^*\} \cong \mathbb{C}^*$
$y \neq 0, \infty$	Δ_y	$ORB_X(y, 0) = T \cdot (a, b)$ $= V(\alpha b^3 - a^2) \setminus \{0\}$

QUESTION $\mathrm{ORB}_X(y_1, \mathcal{F}_1) \overset{?}{=} \mathrm{ORB}_X(y_2, \mathcal{F}_2)$

$$\left\{ \begin{matrix} \text{FACES} \\ \text{OF } \Delta_y \end{matrix} \right\} \xleftrightarrow[\text{ORDER-REVERSING}]{1:1} \left\{ \begin{matrix} \text{CONES} \\ \text{OF } \Lambda_y \end{matrix} \right\}, \quad \mathcal{F} \longmapsto \lambda(\mathcal{F})$$

ANSWER "=" $\Longleftrightarrow$ 1) $\lambda(\mathcal{F}_1) = \lambda(\mathcal{F}_2) \subseteq M_{\mathbb{Q}}$

2) $v_u(y_1) = v_u(y_2)$ FOR SOME $u \in \mathrm{RELINT}(\lambda(\mathcal{F}_i))$

WHERE $v_u : Y \longrightarrow Y_u = \mathrm{PROJ}\left(\bigoplus_{n \geq 0} \Gamma(Y, \mathcal{O}(\mathcal{D}(u)))\right)$

T-ORBIT CLOSURES OF $\tilde{X}$

$$\overline{\mathrm{ORB}_{\tilde{X}}(y, \mathcal{F})} = \mathrm{SPEC}\left(\mathbb{C}[S_y \cap \lambda(\mathcal{F})]\right),$$

WHERE $S_y = \{ u \in \sigma^\vee \cap M \mid \mathcal{D}(u) \text{ PRINCIPAL AT } y \}$

FIBER MONOID COMPLEX

ESTABLISHING A CONEWISE VARYING LATTICE STRUCTURE

TORIC PICTURE

$$\overline{\mathrm{ORB}_{\tilde{X}}(y, \mathcal{F})} = TV\left(\mathbb{Q}_{\geq 0}(\Delta_y - \mathcal{F}) \big/ \mathrm{LIN}(\mathcal{F}) \right) \Big/ M_{y,\mathcal{F}}$$

FINITE GROUP

WHERE $M_{y,\mathcal{F}} = \left(\mathrm{LIN}(\lambda(\mathcal{F})) \cap M\right) \big/ M_{y,\lambda(\mathcal{F})}$

LATTICE OF FULL RANK IN $\mathrm{LIN}(\lambda(\mathcal{F}))$ SPANNED BY $S_y \cap \lambda(\mathcal{F})$

EXAMPLE | T-ORBIT CLOSURES OF $\tilde{X}$

$$\mathcal{D}(u \geq 0) = -\frac{u}{3}\{0\} + \frac{u}{2}\{\infty\}$$

$y \in \mathbb{P}^1$	S_y	$M_{y,\lambda(F)}$ WITH $\dim(F)=0$
0	$S_0 = 3\mathbb{N}$	$\langle S_0 \cap \lambda(-\frac{1}{3})\rangle = 3\mathbb{Z}$
∞	$S_\infty = 2\mathbb{N}$	$\langle S_\infty \cap \lambda(\frac{1}{2})\rangle = 2\mathbb{Z}$
$y \neq 0,\infty$	$S_y = \mathbb{N}$	$\langle S_y \cap \lambda(0)\rangle = \mathbb{Z}$

$$\overline{\mathrm{ORB}_{\tilde{X}}(0,-\tfrac{1}{3})} = \mathrm{TV}\left({Q_{\geq 0}(\Delta_0 - (-\tfrac{1}{3}))} \big/ {\mathrm{LIN}(-\tfrac{1}{3})} \right) \big/ M_{0,-\frac{1}{3}}$$

$$= \mathrm{TV}(\sigma) \big/ \mathbb{Z}/3\mathbb{Z}$$

$$= \mathbb{C} \big/ \mathbb{Z}/3\mathbb{Z}$$

$$\overline{\mathrm{ORB}_{\tilde{X}}(\infty,\tfrac{1}{2})} = \mathbb{C} \big/ \mathbb{Z}/2\mathbb{Z}$$

QUESTION $\overline{\mathrm{ORB}_X(y,\mathcal{F})}$?

[ANSWER] $\mathrm{SPEC}\left(\mathbb{C}\left[G_y \cap \lambda(\mathcal{F})\right]\right)$,

WHERE $G_y = \{ u \in S_y \mid T(y, \Theta(\mathcal{D}(u))) \text{ GEN. } \mathcal{D}(u) \text{ IN } y \}$