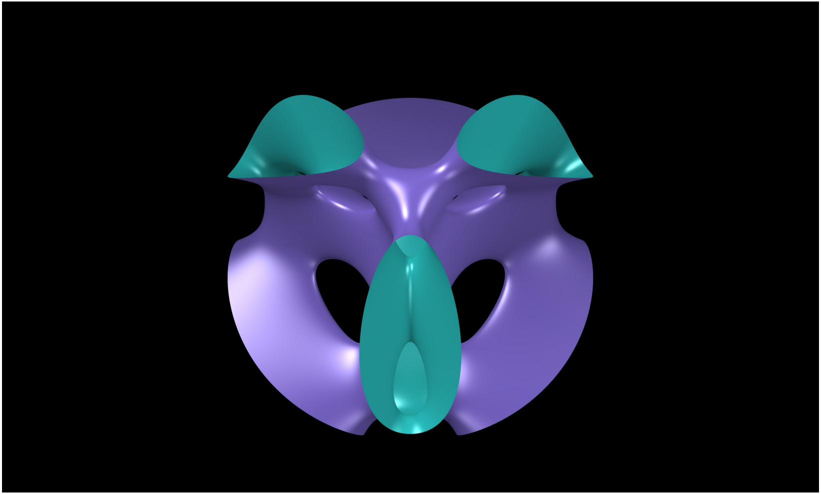
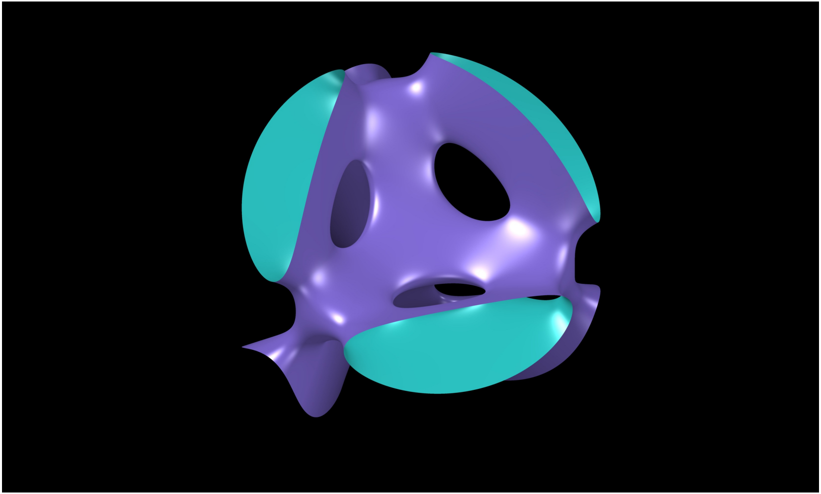


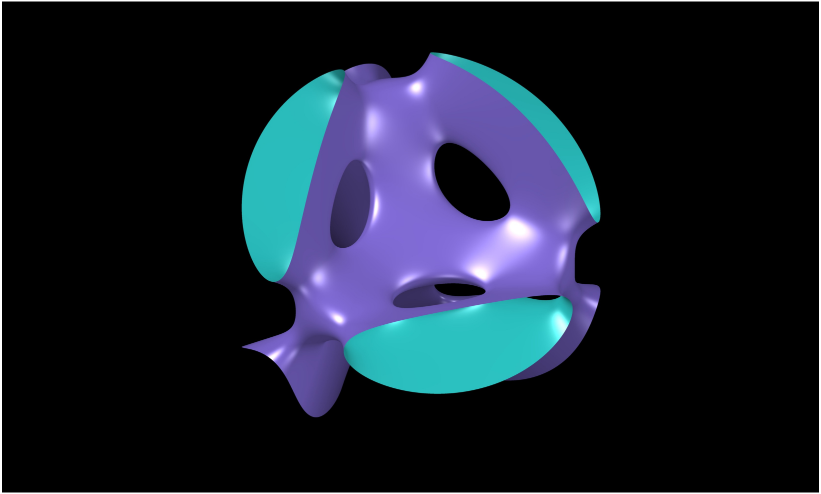
Polytopes, periods, degenerations

Helge Ruddat

Univ. Hamburg & JGU Mainz

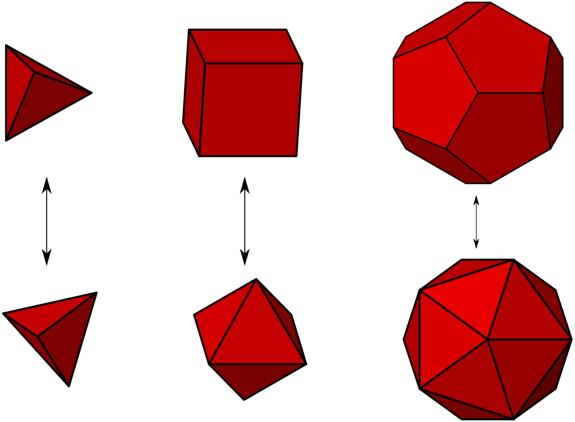


Kummer studied this degeneration of the quartic, with 16 nodes



Today, we consider a yet more serious degeneration:
to a union of planes forming a tetrahedron

The more degenerate, the more combinatorial



Polar duality of polytopes

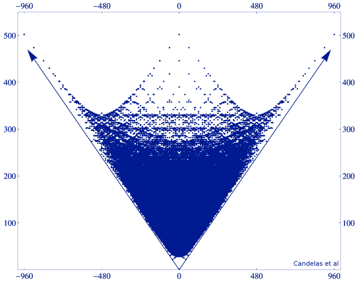
Mirror Symmetry

a) Calabi-Yau manifolds are the higher dimensional version of Kummer's surfaces.

b) When plotting

Euler number
vs
total rank of cohomology

for complex 3-dimensional Calabi-Yau hypersurfaces, mathematical physicists found this diagram.

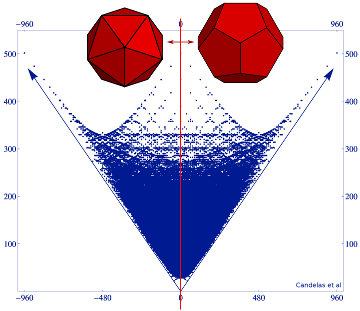


c) The observable symmetry in the diagram is referred to as mirror symmetry.

Mirror Symmetry meets geometry

a) Batyrev discovered ('92) that polar duality explains the symmetry in the diagram.

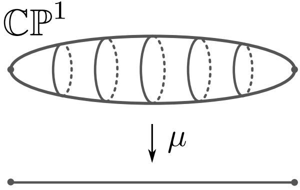
b) Strominger-Yau-Zaslow ('96) proposed that more generally mirror symmetry is explained by a duality of torus fibrations.

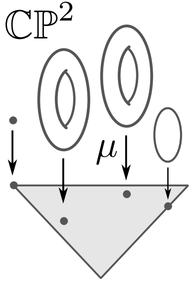


c) Zharkov and Gross-Siebert found a way to bring polar duality and dual torus fibrations together using degenerations.

Every projective toric variety permits a continuous surjection to a polytope.

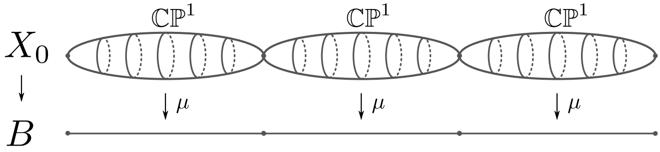
$$\mu : z \mapsto \frac{|z|}{1+|z|} \qquad\qquad \mu : (z_1, z_2) \mapsto \frac{(|z_1|, |z_2|)}{1+|z_1|+|z_2|}$$





Conversely, every polytope with rational vertices gives rise to a projective toric variety.

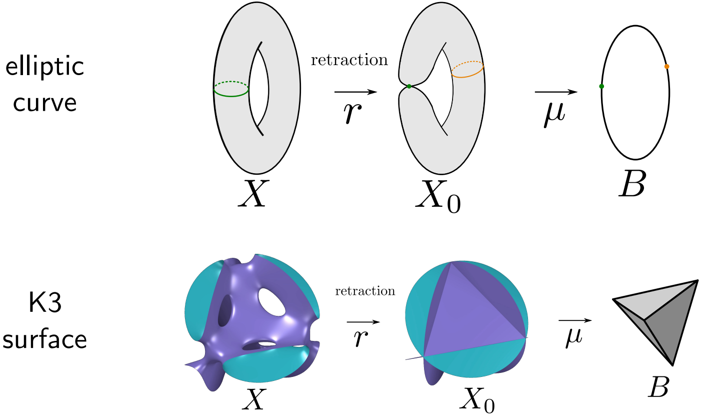
If toric varieties meet in toric boundary strata, we can glue the moment maps.



In particular, we obtain a *dictionary*:

$$\left\{ \begin{array}{c} \text{degenerate manifolds } X_0 \\ \text{consisting of toric varieties} \\ \text{that meet in toric strata} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{topological manifolds } B \\ \text{glued from polyhedra} \end{array} \right\}$$

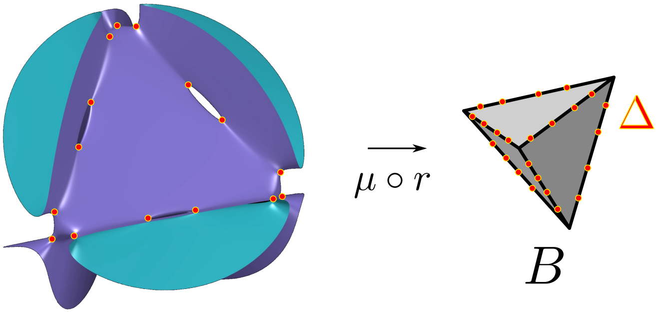
If a manifold X degenerates into X_0, we may compose a retraction map $r : X \to X_0$ with the moment maps $\mu : X_0 \to B$.



The composition $\mu \circ r$ is called the

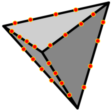
topological Strominger-Yau-Zaslow torus fibration.

If $\dim_{\mathbb{C}} X \geq 2$, there are typically singular torus fibres in the fibration.

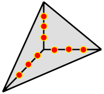


The image of the singular fibres in B is the discriminant Δ in the fibration.

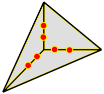
$$XYZW = t \cdot F_4(X, Y, Z, W)$$



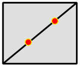
$$XYZ = t \cdot F_3(X, Y, Z, W)$$



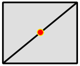
$$XYZ = t \cdot T \cdot F_2(X, Y, Z, W)$$



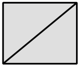
$$XY = t \cdot F_2(X, Y, Z, W)$$

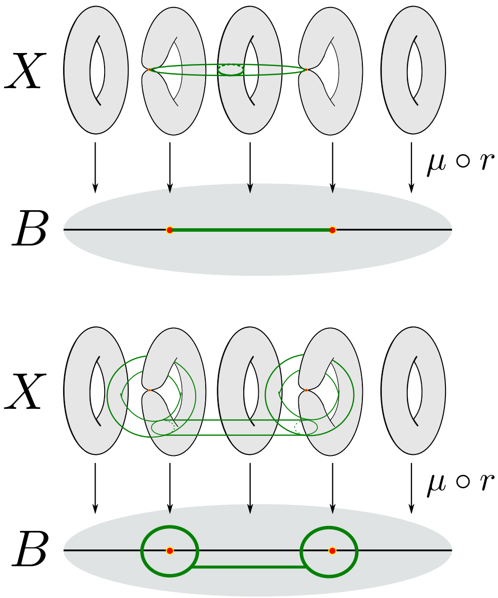


$$XY = t \cdot T \cdot F_1(X, Y, Z, W)$$



$$XY = t \cdot TW$$



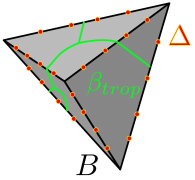

X
μ ∘ r
B
X
μ ∘ r
B

The complement $B \setminus \Delta$ carries an *integral affine structure*, in particular a $\mathbb{Z}^n$-local system of integral tangent vectors Λ.

Let $\iota : B \setminus \Delta \to B$ be the open inclusion and $\iota_*\Lambda$ the pushforward sheaf on B.

Definition

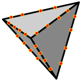
A tropical 1-cycle in B is a singular 1-cycle β_{trop} with coefficients in $\iota_*\Lambda$. The homology group is denoted $H_1(B, \iota_*\Lambda)$.

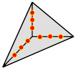


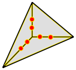
Lemma

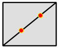
There is a natural homomorphism

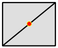
$$H_1(B, \iota_*\Lambda) \to H_n(X, \mathbb{Z})/\mathbb{Z}(\textit{SYZ-fiber}) \qquad \beta_{trop} \mapsto \beta.$$

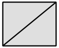
$XYZW = t \cdot F_4(X,Y,Z,W)$  $\operatorname{rank}(H_1(B, \iota_* \Lambda)) = 20$

$XYZ = t \cdot F_3(X,Y,Z,W)$  $\operatorname{rank}(H_1(B, \iota_* \Lambda)) = 7$

$XYZ = t \cdot T \cdot F_2(X,Y,Z,W)$  $\operatorname{rank}(H_1(B, \iota_* \Lambda)) = 4$

$XY = t \cdot F_2(X,Y,Z,W)$  $\operatorname{rank}(H_1(B, \iota_* \Lambda)) = 1$

$XY = t \cdot T \cdot F_1(X,Y,Z,W)$  $\operatorname{rank}(H_1(B, \iota_* \Lambda)) = 0$

$XY = t \cdot TW$  $\operatorname{rank}(H_1(B, \iota_* \Lambda)) = 0$

Let $\check{\Lambda} := \text{Hom}(\Lambda, \mathbb{Z})$ denote the local system dual to Λ.

Theorem (R. 2020, to appear in Geom. Topol.)

There is a natural pairing

$$H_1(B, \iota_*\Lambda) \otimes H^1(B, \iota_*\check{\Lambda}) \to \mathbb{Q}. \qquad (1)$$

which is perfect if the discriminant Δ is symple.

For $\dim B = 2$, symple means that the monodromy around each point is $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ for some $k \geq 0$.

For $\dim B = 3$, symple means that every vertex of Δ is trivalent and the monodromy around each edge of Δ is like in the product of the symple situation in dimension two plus a trival $\mathbb{R}$-factor.

For $\dim B = 3$, the pairing is not perfect if Δ has a fourvalent point (conifold).

Corollary

In the symple case, the natural homomorphism

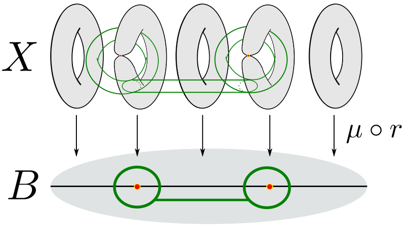
$$H_1(B, \iota_*\Lambda) \to H_n(X, \mathbb{Z})/\mathbb{Z}(\textit{SYZ-fiber}) \qquad \beta_{trop} \mapsto \beta$$

is injective.

Example

For the degeneration of a quadric with log Calabi-Yau boundary D a union of two $\mathbb{P}^1$

$$XY = tF_2(X, Y, Z, W)$$



The map $H_1(B, \iota_*\Lambda) \to H_2(X \setminus D, \mathbb{Z})/\mathbb{Z}(\text{SYZ-fiber})$ is an isomorphism over $\mathbb{Q}$.

Definition

A *period integral* $\int_\beta \Omega$ is the integral of a holomorphic differential n-form Ω on a complex manifold X over an n-cycle $\beta \in H_n(X, \mathbb{Z})$.

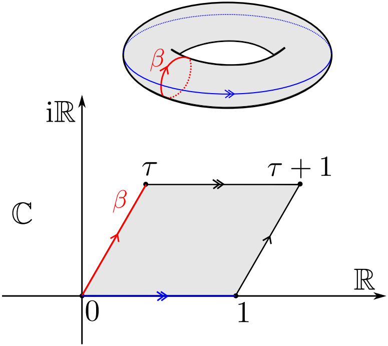
Example

For the elliptic curve

$$X = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau),$$

the parameter τ is a period integral because for $\Omega = dz$,

$$\oint_\beta dz = \int_0^\tau dz = \Big[z\Big]_0^\tau = \tau.$$



A Calabi-Yau n-manifold has a unique (up to scale) differential n-form Ω. It remains to think about what β to integrate over.

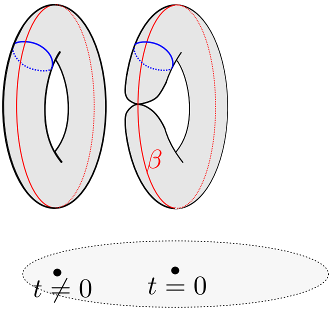
Example

The Tate family is obtained by applying the exponential map

$$\begin{array}{rcl} X = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) & \to & \mathbb{C}^*/t^{\mathbb{Z}} =: E_t \\ z & \mapsto & \exp(2\pi i z) \end{array}$$

for $t = exp(2\pi i \tau)$ in the punctured unit disk.

This family E_t can be extended over $t = 0$ with a degeneration.



What happens to the period integral under degeneration?

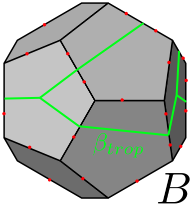
Observe that

$$\oint_\beta dz = \log t,$$

so we get a log pole!

Theorem (R.-Siebert, Publ. math. IHÉS 132 (2020))

Let β_{trop} be a tropical 1-cycle in the intersection complex B of a degenerate Calabi-Yau n-fold X_0. Let $\beta \in H_n(X, \mathbb{Z})$ denote the natural associated n-cycle in X. The period integral $\int_\beta \Omega$ is well-defined even over Artin rings supported at $t = 0$ and is computed by the formula



$$\frac{1}{(2\pi i)^{n-1}} \int_\beta \Omega = \kappa \cdot \log t + \textit{(glueing term)} + \textit{(Ronkin-term)}$$

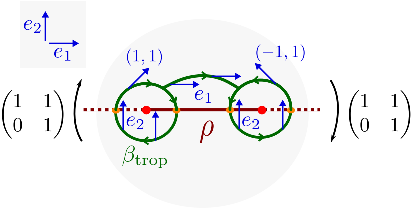
for $\kappa \in \mathbb{Z}$ the number of crossings of β_{trop} with codimension one walls in B and t the smoothing parameter.

Example

Fix $a, b \in \mathbb{C}$ and consider the degeneration

$$xy = t(au^{-1} + 1)(1 + bu).$$

The dual intersection complex B is



The exponentiated period integral for the green tropical cycle is

$$\exp\left(\frac{1}{(2\pi i)^{n-1}} \int_\beta \Omega\right) = a \cdot b.$$

Theorem (Gross and Siebert, 2011)

If X_0 is obtained from a B with simple discriminant, then there is a canonical formal smoothing of X_0. The formal family is defined over

$$\mathbb{C}[H^1(B, \iota_*\check{\Lambda})^*][\![t]\!].$$

Theorem (R.-Siebert)

The Ronkin term vanishes for Gross-Siebert formal families and the period integral for β_{trop} is computed via the map

$$H_1(B, \iota_*\Lambda) \to (H^1(B, \iota_*\check{\Lambda}))^*, \qquad \beta_{trop} \mapsto \beta_{trop}^{**}$$

coming from the pairing $H_1(B, \iota_*\Lambda) \otimes H^1(B, \iota_*\check{\Lambda}) \to \mathbb{Z}$, namely

$$\exp\left(\frac{1}{(2\pi i)^{n-1}} \int_\beta \Omega\right) = t^{\langle \beta_{trop}, c_1(\varphi)\rangle} z^{\beta_{trop}^{**}}$$

where $c_1(\varphi) \in H^1(B, \iota_*\check{\Lambda})$ is the class of a multivalued piecewise linear function on B that is used to construct the formal family.

Theorem (R.-Siebert)

The formal Gross-Siebert family is semi-universal, paramatrized in canonical coordinates and analytifies to a holomorphic family.

We have seen mirror symmetry as an equality of Hodge numbers earlier in the talk, in particular

$$H^1(X, \Omega_X) \cong H^1(\check{X}, \Theta_{\check{X}})$$

holds for a mirror pair $(X, \check{X})$. The versal Gross-Siebert family is literally defined over $H^1(\check{X}, \Theta_{\check{X}}) = H^1(B, \iota_* \check{\Lambda}) \otimes \mathbb{C}$ with exponentiated period integrals being monomials in the natural linear coordinates. Under *discrete Legendre transform* (a generalization of polar duality), the mirror dually constructed family $\check{X}$ satisfies

$$H^1(B, \iota_* \check{\Lambda}) \otimes \mathbb{C} = H^1(\check{X}, \Omega_{\check{X}})$$

by a *canonical isomorphism*.

Thank you!