"COMPUTATIONAL ASPECTS OF ORBIFOLD EQUIVALENCE"

ONLINE ALGEBRAIC GEOMETRY SEMINAR

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⓪ Motivation + plans

Main character today:

Landau-Ginzburg models

Origin:
superconductivity!
~1950

[Vafa-Warner, '88]:
update to string theory!
(2,2)-supersymmetric sigma model characterized by some function W

Famous not only because of string theory but also
- mirror symmetry!

⇒ a way to relate them:

orbifold equivalence!

Plan 4 today:

1. Intro 2 orbifold equivalences
2. Examples of orbifold equivalence
3. Open questions re. orb. equiv.

1 Introduction to orbifold equivalence

Let $k(=\mathbb{C})$ field, $S := k[x_1, \dots, x_m]$, $W \in S$. Assign $|x_i| \in \mathbb{Q}_{\geq 0}$ to each x_i.

Defn: • W **potential** $:\Longleftrightarrow \dim_k \left(S / \langle \partial_{x_1} W, \dots, \partial_{x_m} W \rangle \right) < \infty$.

(Alternatively: W has an isolated singularity at $\underline{0}$).

Examples: $S = \mathbb{C}[x]$, $W = x^m$

$S = \mathbb{C}[x, y, z]$, $W = x^4 z + y^3 + z^2$. (E_{14})

• A potential W is **homogeneous of degree d** $\in \mathbb{Q}_{\geq 0}$ if in addition it satisfies that: $W(\lambda^{|x_1|} x_1, \dots, \lambda^{|x_m|} x_m) = \lambda^d W(x_1, \dots, x_m)$ $\forall \lambda \in \mathbb{C}^\times$.

Example: (follow-up) if $|x| = \frac{d}{m}$, $W(\lambda^{d/m} x) = (\lambda^{d/m} x)^m = \lambda^d x^m = \lambda^d W(x)$.

• **Central charge of a potential W**: $c_W := \sum_{i=1}^{m} (1 - |x_i|)$

Fix: "potential" = "homogeneous potential of degree 2"

Let us also set up the notation:

$\mathcal{P}_k := \{$ set of potentials with coefficients in k, and any number of variables $\}$

GOAL TODAY: define an equivalence relation in $\mathcal{P}_k$!

How? Matrix factorizations!

Defn: given (S_1, W_1), (S_2, W_2) pairs of a polynomial ring + potential, a **matrix factorization** of $W_1 . \mathrm{id} - \mathrm{id} . W_2$ (short: $W_1 - W_2$) consists of a pair (M, d^M) where:

- M free, $\mathbb{Z}_2$-graded $(= M_0 \oplus M_1)$ finite rank S_1-S_2-bimodule,
- $d^M : M \to M$ degree 1 $\left(= \begin{pmatrix} 0 & d_1^M \\ d_0^M & 0 \end{pmatrix} \right)$ S_1-S_2-linear endomorphism

such that: $d^M \circ d^M = W_1 . \mathrm{id}_M - \mathrm{id}_M . W_2$ ("twisted differential").

Ex: $(S_1, W_1) = (\mathbb{C}[x], x^d)$
$(S_2, W_2) = (\mathbb{C}[y], y^d)$ $\rightsquigarrow$ $\left(\mathbb{C}[x,y]^{\oplus 2}, \begin{pmatrix} 0 & x-y \\ \frac{x^d - y^d}{x-y} & 0 \end{pmatrix} \right) =: I$

Rmk: definition tailored to our purposes, can be modified in several ways!

Defn: given two matrix factorizations (M, d^M), (N, d^N), a **morphism of matrix factorizations** is an S-bilinear map $f : M \to N$.

Hopefully enough to construct a category...!

Define: $MF(W_1 - W_2) := \begin{cases} \text{ob: matrix factorizations of } W_1.\mathrm{id} - \mathrm{id}.W_2 \\ \text{mor: morphisms of matrix fact's.} \end{cases}$

Fact: $MF(W_1 - W_2)$ has the structure of a differential $\mathbb{Z}_2$-graded category, with a differential at the morphism space:

$$\delta(f) = d^N \circ f - (-1)^{|f|} f \circ d^M \qquad \text{for } f \in \mathrm{Mor}_{MF(W_1 - W_2)}((M, d^M), (N, d^N))$$

Using this structure, we define a much more interesting subcategory:

$$HMF(W_1 - W_2) := H^0(MF(W_1 - W_2))$$

$$= \begin{cases} \text{ob: same as } \mathrm{Ob}(MF(W_1 - W_2)) \\ \text{mor: } \dfrac{\{ f \in \mathrm{Mor}_{MF(W_1 - W_2)}((M, d^M), (N, d^N)) \text{ with } |f| = 0 \mid \delta(f) = 0 \}}{\{ f \in \text{ — '' — with } |f| = 1 \mid \mathrm{Im}(\delta(f)) \}} \end{cases}$$

Why?

Thm: [Orlov, '05] $HMF(W)$ is equivalent to $D^b(\mathrm{Coh}(X))$, the bounded derived category of coherent sheaves over X, which is defined by $W = 0$.

Ain't it cool ☺

Another interesting structure in HMF ...

Defn: given $(S_1, W_1), (S_2, W_2), (S_3, W_3)$,

(M, d^M) matrix factorization of $W_1 - W_2$

(N, d^N) matrix factorization of $W_2 - W_3$,

the tensor product matrix factorization $(M \otimes_{S_2} N, d^{M\otimes N})$ is the matrix factorization of $W_1 - W_3$ with:

- $M \otimes_{S_2} N$ S_1-S_3-bimodule,
- $d^{M\otimes N} := d^M \otimes id_N + id_M \otimes d^N$.

Rmk: when composing $\otimes$'s of graded morphisms, don't forget to use the koszul sign rule!

Thm:
- [Carqueville - Runkel, '09] HMF$(W.id - id.W)$ is a monoidal cat.
- [Carqueville - Murfet, '12] every object in HMF has a left and a right adjoint ("categorical duals") for which we have very explicit expressions.

Which is great for our purposes...

Out of the categorical duals, we can define some interesting morphisms:

Defn/Prop: let $V(x_1,\dots,x_m)$, $W(y_1,\dots,y_m) \in \mathcal{P}_k$,
(M, d^M) matrix factorization of $W-V$.

(Up to a sign,) Assign to (M,d^M) a left (right) quantum dimension,

$$\mathrm{qdim}_e(M,d^M) = \mathrm{Res}\left[\frac{\mathrm{str}(\partial x_1 d^M \cdots \partial x_m d^M \, \partial y_1 d^M \cdots \partial y_m d^M)\, \underline{dy}}{\partial y_1 W, \dots, \partial y_m W}\right]$$

$$\left(\mathrm{qdim}_r(M,d^M) = \mathrm{Res}\left[\frac{\text{—}\,''\,\text{—} \qquad \underline{dx}}{\partial x_1 V, \dots, \partial x_m V}\right]\right)$$

We say that V and W are orbifold equivalent if $\exists$ such a (M,d^M) for which both qdim_e and qdim_r are invertible. Notation: $V \sim_{orb} W$

Orbifold equivalence is indeed an equivalence relation in $\mathcal{P}_k$.

Rmk:
- $\mathrm{qdim}_{e,r} \in S$ (and if $\mathbb{Q}$-grading, $\in k$).
- Note that if $W \sim_{orb} V \Rightarrow c_W = c_V$.
  Converse not expected to be true.
- a way to see this relation:

$$D^b(\mathrm{Coh}(X_W)) \xrightarrow{-\otimes(M,d^M)} D^b(\mathrm{Coh}(X_V))$$

!

Ex: $x^d \sim_{ab} y^d$, via I!

Let's compute this example in detail: here, $W - V = x^d - y^d$

a) $$\partial_x d^I \, \partial_y d^I = \begin{pmatrix} 0 & 1 \\ (d-1)x^{d-2} + (d-2)x^{d-3}y + \dots + 2xy^{d-3} + y^{d-2} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ (d-1)y^{d-2} + (d-2)y^{d-3}x + \dots + 2yx^{d-3} + x^{d-2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \color{purple}\bullet & 0 \\ 0 & -\color{green}\bullet \end{pmatrix}$$

b) $$\mathrm{str}(\partial_x d^I \, \partial_y d^I) = \color{purple}\bullet \color{black} - [-\color{green}\bullet\color{black}] = d[x^{d-2} + x^{d-3}y + \dots + xy^{d-3} + y^{d-2}]$$

c) Residue? Need the coefficient of the $1/x$ term in

↳ qdim_l: $$\frac{1}{2\pi i} \oint \frac{\mathrm{str}(\partial_x d^I \, \partial_y d^I)}{d\,x^{d-1}} \rightsquigarrow 1$$

↳ qdim_r: $$\frac{1}{2\pi i} \oint \frac{\mathrm{str}(\partial_x d^I \, \partial_y d^I)}{-d\,y^{d-1}} \rightsquigarrow -1.$$

Hey! This is like, the easiest... ~U

(2) Some examples of orbifold equivalence

[Arnold, ~70s]

Modality	Possible orbifold equivalences	Proven in ...
Simple	$W_{A_{d-1}} \sim_{orb} W_{D_{\frac{d}{2}+1}}$, d even $\neq 12, 18, 30$	[Carqueville-Runkel-RC, '13]
	$W_{A_{11}} \sim_{orb} W_{D_7} \sim_{orb} W_{E_6}$	-"-
	$W_{A_{17}} \sim_{orb} W_{D_{10}} \sim_{orb} W_{E_7}$	-"-
	$W_{A_{29}} \sim_{orb} W_{D_{16}} \sim_{orb} W_{E_8}$	-"-
Unimodal of exceptional type	$W_{Q_{10}} \sim_{orb} W_{E_{14}}$	[Newton-RC, '15] (+ autoequivalences in [Newton-RC, '16])
	$W_{Q_{11}} \sim_{orb} W_{Z_{13}}$	
	$W_{S_{11}} \sim_{orb} W_{W_{13}}$	[Recknagel et al, '17]
	$W_{Z_{11}} \sim_{orb} W_{E_{13}}$	
Bimodal	$W_{Q_{18}} \sim_{orb} W_{E_{30}}$	[Kluck-RC, '19]
	$W_{E_{18}} \sim_{orb} W_{Q_{12}}$	
	... (7 more)	

Rmk: • simple sing's: in two variables,

$$W_{A_{d-1}} = x^d + y^2 \qquad W_{E_6} = x^3 + y^4 \qquad W_{E_8} = x^3 + y^5$$

$$W_{D_{d+1}} = x^d + xy^2 \qquad W_{E_7} = x^3 + xy^3$$

• unimodal: these singularities are related by strange duality,

$\Rightarrow$ [Martinec]: $c_W = \dfrac{h+2}{h}$, where h Coxeter number of the sing.

Further, [Ebeling-Takahashi] they display mirror-symmetric behaviour. In particular, some are Berglund-Hübsch mirror pairs!

E.g. $W_{Q_{10}} = x^4 + y^3 + xz^2$

$$W_{E_{14}} = \begin{cases} x^4 z + y^3 + z^2 \\ x^8 + y^3 + z^2 \end{cases}$$

• [Newton-RC]: Galois groups control (some of) the orbifold equivalences.

E.g. (follow-up) $W_{Q_{10}} \sim_{orb} W_{E_{14}}$ has $D_8 \times C_2$

$W_{Q_{10}} \sim_{orb} W_{E_{14}}$ has V_4

This search is actually quite a heavy computational problem.

– Is there any way out?

Prop: [Kreuck-RC; 19] $\exists$ an algorithm to prove orbifold equivalence that terminates if the two potentials are equivalent.

But – on a reasonable amount of time? A comparison from cryptography...

	Us ($Q_{10} \sim ab\ E_{14}$)	Fukuoka MQ Challenge
# indeterminates	108	74
# equations	237	148
Base field	$\mathbb{C}$	Field of... 2 elements!

Quite sobering. But there's hope!

$\rightsquigarrow$ "lucky guessing strategy" [Kreuck-RC, RC in progress]

$\rightsquigarrow$ [Folsom-Williams-RC, in progress]: implementing some machine learning techniques to simplify the search.

More soon! :)

③ Some open questions re orbifold equivalence

There's still plenty to learn out of these equivalences...

a) Remaining conjectures for bimodal singularities?

b) General decidability method?

c) Role of Galois groups? Can we count orbifold eq's?

d) Applications (to Landau-Ginzburg / conformal field theory correspondence)?

e) ...

THANK YOU VERY MUCH
FOR YOUR ATTENTION ! :)

Questions?

Maybe later?

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