Wall-crossing does not induce MMP

Fatemeh Rezaee

Plan of the talk

- Main goal
- Some definitions/notations
- State the problem/Motivation
- History
  - Surfaces
- Bridgeland stability conditions
- Back to the problem
- Main Theorem
- Idea of the proof
- More pictures

Main goal

Description of a (birationally) interesting wall-crossing

$\rightsquigarrow$ Wall-crossing can be more complicated than was previously known.

$\rightsquigarrow$ Failure of the wall-crossing/MMP correspondence.

(Big picture: description of some classical moduli spaces)

Notation/Definition

X a smooth projective variety. v: a class (e.g. Chern character).

- An object $E \in D^b(X) = D^b(Coh(X))$ is *(semi)-stable* with respect to the "slope" μ, if $\mu(F)(\leq) < \mu(E)$ fo any sub-object $F \subset E$.
- $Stab(X)$: stability "manifold" of all stability conditions on X, [Bridgeland] (sometimes, we call it *stability space*)
- $\mathcal{M}_\sigma(v)$: space of σ-stable objects of class v in $D^b(X)$.
- For a non-singular 3-fold X, we define a *Pandharipande-Thomas stable pair* $(\mathcal{F}, s)$ where $\mathcal{F}$ is a sheaf supported on curves in X with zero-dimensional cokernel of the sections $s\colon \mathcal{O}_X \to \mathcal{F}$.
- For a category $\mathcal{C}$, we define the *Grothendieck group $K_0(\mathcal{C})$* to be a free abelian group (usually not f.g.) generated by the objects in $\mathcal{C}$ with relations $A + B = E$ for any short exact sequence $A \to E \to B$.

Minimal Model Program (MMP)

Let M be a smooth projective variety.

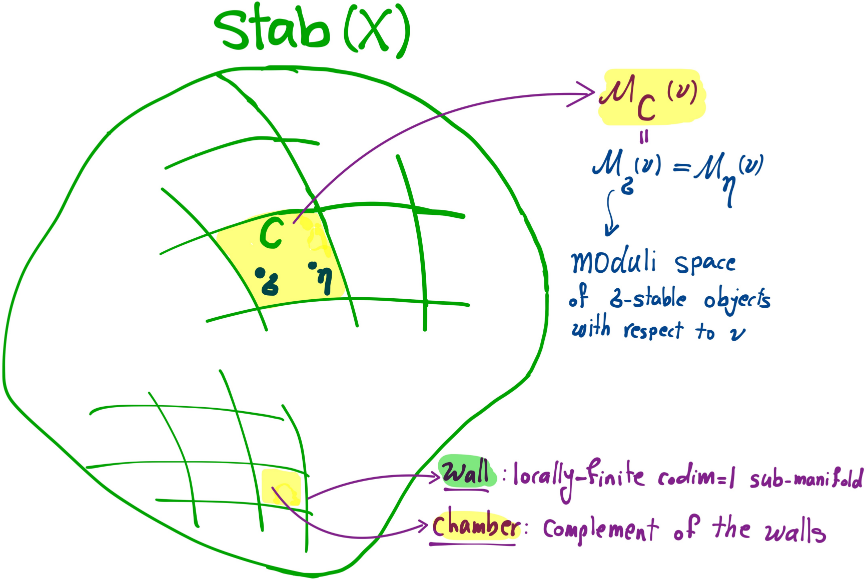
Definition. A *Minimal Model Program (MMP)* is a sequence of divisorial contractions or flips

$$M = M_0 \dashrightarrow M_1 \dashrightarrow M_2 \dashrightarrow \ldots\ldots\ldots\ldots \dashrightarrow M_N$$

such that each M_i is at least $\mathbb{Q}-$ *factorial* (i.e. any Weil divisor is $\mathbb{Q}-$ *Cartier*) and M_N is either a minimal model (K_{M_N} is nef) or has a Mori fiber space structure.

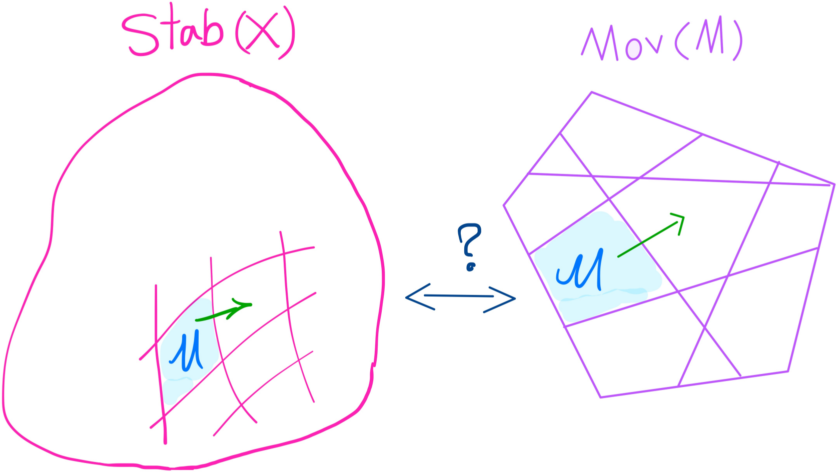
We refer to each step in the sequence as "MMP step".

Stability manifold and wall-chamber decomposition



Wall-Crossing/MMP correspondence

Let X be a variety, and $M = \mathcal{M}_\sigma(X)$ the moduli space of stable objects associated to a chamber in $Stab(X)$.



Question

Is there a correspondence between the Bridgeland wall-crossing in $Stab(X)$ and the Mori wall-crossing in $\mathrm{Mov}(M)$?

Surfaces

The answer is affirmative for most of the cases :

(some examples:)

- **X=K3 surface** [Bayer-Macrì('14)]
- **X=**$\mathbb{P}^2$ [Arcara-Bertram-Coskun-Huizenga('13); Bertram-Martinez-Wang ('14); Li-Zhao ('18)]
- **X=Enriques Surface** [Neur-Yoshioka('19); Beckmann('20)]
- **M=Smooth projective surface** [Toda('13)]

MMP/Wall-crossing correspondence on surfaces

S: K3 surface, and v a primitive class.

Theorem [Bayer-Macrì] Let σ, δ be generic stability conditions with respect to v. Then the two moduli spaces $\mathcal{M}_\sigma(v)$ and $\mathcal{M}_\delta(v)$ of Bridgeland-stable objects are birational to each other.

*Identify the Néron-Severi groups of $\mathcal{M}_\sigma(v)$ and $\mathcal{M}_\delta(v)$.

*C a chamber; the main result of [Bayer-Macrì] gives a natural map

$$l_C \colon C \to NS(\mathcal{M}_C(v))$$

to the Néron-Severi group of the moduli space, whose image is contained in the ample cone of $\mathcal{M}_C(v)$.

(MMP/Wall-crossing correspondence on surfaces)

Theorem [Bayer-Macrì] Fix a base point $\sigma \in \mathit{Stab}(S)$.
(a) Under the identification of the Néron-Severi groups, the maps l_C glue to a piece-wise analytic continuous map

$$L\colon \mathit{Stab}(S) \to NS(\mathcal{M}_\sigma(v)).$$

(b) The map L is compatible, in the sense that for any generic $\sigma' \in \mathit{Stab}(S)$, the moduli space $\mathcal{M}_{\sigma'}(v)$ is the birational model corresponding to $L(\sigma')$. In particular, every smooth K-trivial birational model of $\mathcal{M}_\sigma(v)$ appears as a moduli space $\mathcal{M}_C(v)$ of Bridgeland stable objects for some chamber $C \subset \mathit{Stab}(S)$.

*Part (b) says MMP can be run via wall-crossing:
Any birational model can be reached after wall-crossing as a moduli space of stable objects.

threefolds

For $\mathbb{P}^3$:

- For some cases, the answer is "partially" affirmative.
  - **Hilbert scheme of twisted cubics in** $\mathbb{P}^3$
    [Schmidt (2015); Xia (2016)]
  - **Hilbert scheme of elliptic quartics in** $\mathbb{P}^3$
    [Gallardo-Huerta-Schmidt(2016)]

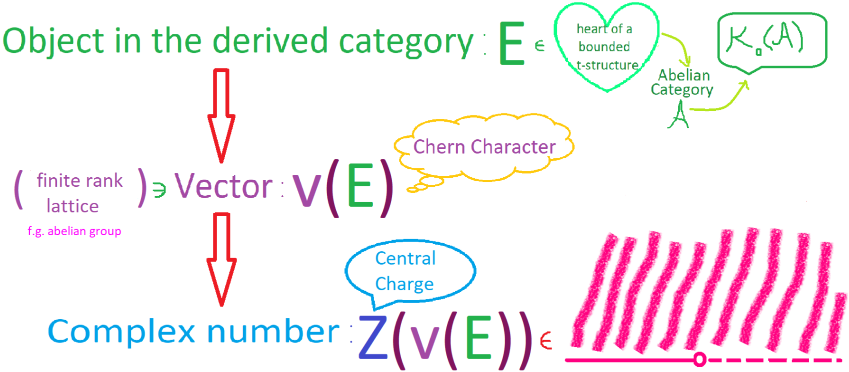
  * Both Hilbert schemes have 2 irreducible components.

  * Wall-Crossing $\Rightarrow$ *MMP*

  * Wall-Crossing $\not\Leftarrow$ *MMP*

- We exhibit an example for which both directions are false.

Very rough idea of "Bridgeland stability conditions":



Then we can define "slope(E)" : $= \frac{-Re(Z(v(E)))}{Im(Z(v(E)))}$

$\rightsquigarrow$ compare the slopes and define (semi-)stability.

Stability on abelian categoties

$\mathcal{A}$ an abelian category. A pair $(\mathcal{A}, Z)$ is *stability conditions* if Z is a group homomorphism, called a *central charge* $Z\colon K_0(\mathcal{A}) \to \mathbb{C}$ where $K_0(\mathcal{A})$ is the Grothendieck group of $\mathcal{A}$, such that

- For each non-zero object E in $\mathcal{A}$, we have $Im(Z(E)) \geq 0$ and if $Im(Z(E)) = 0$, then $Re(Z(E)) < 0$,
- (Harder-Narasimhan filtration) For any non-zero object E in $\mathcal{A}$, there is a filtration
$$0 = E_0 \subset E_1 \subset E_2 \subset ... \subset E_n = E$$
where E_i are objects in $\mathcal{A}$ and $A_i \colon = E_i/E_{i-1}$ are semistable objects with $\mu(A_i) \geq \mu(A_{i-1})$ for each i.

Example. Let C be a projective curve, and define Z as $Z(\mathcal{E})\colon = -deg(\mathcal{E}) + i.rk(\mathcal{E})$, for any object $\mathcal{E}$ in $Coh(C)$. Therefore $(Coh(C), Z)$ defines stability conditions.

Stability conditions on higher dimensional varieties

Issue: We cannot define any central charge for $\mathcal{A} = Coh(X)$ when $dim(X) \geq 2$.

Solution: Try to find another abelian category in $D^b(X)$.

*A *torsion pair* in an abelian category $\mathcal{A}$ is a pair $\mathcal{T}, \mathcal{F}$ of full additive subcategories with (1) $Hom(\mathcal{T}, \mathcal{F}) = 0$. (2) For all $E \in \mathcal{A}$ there exists a short exact sequence $0 \to T \to E \to F \to 0$ where $T \in \mathcal{T}$, $F \in \mathcal{F}$.

*A *heart of a bounded t-structure* $\mathcal{A}$ on $D^b(X)$ is a full additive subcategory of $D^b(X)$ such that

- $Hom(A[i], B[j]) = 0$ for all $A, B \in \mathcal{A}$ and $i > j$.
- Harder-Narasimhan property.

* $\mathcal{A}$ is an abelian category, and $K_0(\mathcal{A}) = K_0(X) = K_0(D^b(X))$.

*Fix a finite rank lattice Λ and a group homomorphism $v\colon K_0(X) \twoheadrightarrow \Lambda$, such that the central charge factor via this morphism.

Bridgeland stability conditions

Let X be a variety of dimension n. A pair $\sigma = (\mathcal{A}, Z)$ is a *Bridgeland stability conditions* on $D^b(X)$ if

- $\mathcal{A}$ is a heart of a bounded t-structure,
- The central charge $Z\colon \Lambda \to \mathbb{C}$, is an additive homomorphism, (Λ finite rank lattice)
- For any non-zero object E in the heart, we have $Z(v(E)) \in \mathbb{H} \cup \mathbb{R}_{<0}$, where $\mathbb{H}$ is the upper half plane in $\mathbb{C}$,
- Support property.

Support property $\rightsquigarrow$ $Stab(X)$ admits a chamber decomposition, depending on v, such that:
(i) for a chamber C, the moduli space $\mathcal{M}_\sigma(v) = \mathcal{M}_C(v)$ is independent of the choice of $\sigma \in C$, and
(ii) walls consist of stability conditions with strictly semistable objects of class v ([Bayer-Macrì]).

Stability conditions on $\mathbb{P}^3$

Bridgeland stability conditions does exist on $\mathbb{P}^3$ ([Macrì], [Bayer-Macrì-Toda], [Bayer-Macrì-Stellari]):

- *Double tilting* $Coh(\mathbb{P}^3) = < \mathcal{T}, \mathcal{F} >$
  $\rightsquigarrow$ new heart of a bounded t-structure
- Central charge
- Support property satisfied

*There exist a wall-chamber structre on $Stab(\mathbb{P}^3)$

Back to the problem/example

Setup

Recall: A smooth non-hyperelliptic genus 4 curve C embeds into $\mathbb{P}^3$ as a (2,3)-complete intersection curve.

Question: How to compactify this 24-dimensional space?

Classical Answer Hilbert scheme of such curves.

However: Many irreducible components.
Hard to even list all the irreducible components!

Instead: Bridgeland stability conditions on $D^b(\mathbb{P}^3)$ give better compactifications, depending on a choice of a stability condition $\sigma \in Stab(\mathbb{P}^3)$ gives $\mathcal{M}_\sigma(1, 0, -6, 15)$, the moduli space of σ-stable complexes E with $Ch(E) = Ch(\mathcal{I}_C)$.

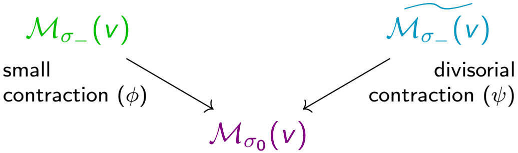
Approach

Following a path along the space of stability conditions to understand how $\mathcal{M}_\sigma(1, 0, -6, 15)$ changes:

- **beginning of the path:** Efficient compactification, given by a $\mathbb{P}^{15}$-bundle (choice of cubic) over $\mathbb{P}^9$ (choice of quadric), parametrising some non-torsion free sheaves in addition to ideal sheaves.
- **Large-volume limit** Recovers the Hilbert scheme.
- **Intermediate step:** moduli space of PT stable pairs.
- **Second wall-crossing:** Detailed analysis of wall-crossing gives novel features, as explained in the following.

Theorem 1 ([R20]). Fix $v = (1, 0, -6, 15)$. There is a wall-crossing $\mathcal{M}_{\sigma_-}(v) \to \mathcal{M}_{\sigma_+}(v)$ such that:

- $\mathcal{M}_{\sigma_-}(v)$ is a smooth and irreducible variety.
- $\mathcal{M}_{\sigma_+}(v) = \widetilde{\mathcal{M}_{\sigma_-}(v)} \cup \mathcal{M}'$, where $\widetilde{\mathcal{M}_{\sigma_-}(v)}$ is birational to $\mathcal{M}_{\sigma_-}(v)$ and $\mathcal{M}'$ is a new irreducible component.
- There is a diagram (where σ_0 is on the wall)



where both ϕ and ψ have relative Picard rank 1. In particular, $\widetilde{\mathcal{M}_{\sigma_-}(v)}$ is not $\mathbb{Q}-$factorial.

How to prove Theorem 1?

The components before and after crossing the wall:

- $\mathcal{M}_{\sigma_-}(v)$: a blow-up of a $\mathbb{P}^{15}$-bundle over $\mathbb{P}^9$
- $\mathcal{M}'$: a $\mathbb{P}^{17}$-bundle over $\mathbb{G}r(2,4) \times \mathfrak{Fl}_2$, where $\mathfrak{Fl}_2$ is the space parametrizing flags $Z_2 \subset P \subset \mathbb{P}^3$ where P is a plane and Z_2 a zero dimensional subscheme of length 2.

Let $\mathcal{W}$ is the wall between $\mathcal{M}_{\sigma_-}(v)$ and $\mathcal{M}_{\sigma_+}(v)$. Then we have $\mathcal{W} = \langle \mathcal{I}_L(-1), \iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5) \rangle$, where L is a line, P a plane, Z_2 a zero-dimensional subscheme of length 2, and $\iota_P \colon P \hookrightarrow \mathbb{P}^3$ is the inclusion map.

ϕ is small

Description of destabilizing locus:

Proposition ([R20]) The destabilizing locus in $\mathcal{M}_{\sigma_-}(v)$ when crossing $\mathcal{W}$ is of dimension 10, and it contains the exceptional locus of $\phi\colon \mathcal{M}_{\sigma_-}(v) \to \mathcal{M}_{\sigma_0}(v)$ of dimension 8 which is a $\mathbb{P}^1$-bundle over its 7-dimensional image under ϕ.

Corollary ([R20]) ϕ is a small contraction.

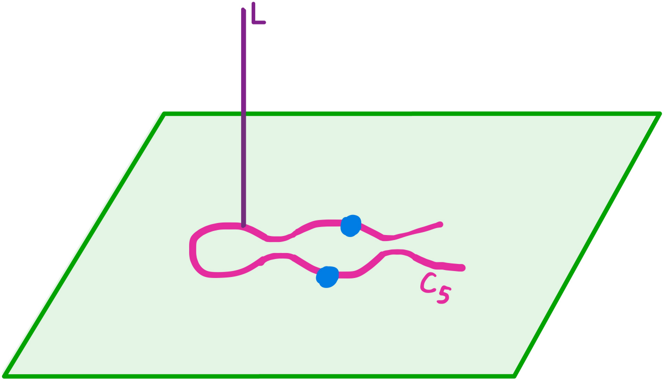
Key step to prove ψ is divisorial

Description of the intersection of the 2 components:

Theorem 2 ([R20]). The intersection $\widetilde{\mathcal{M}_{\sigma_-}(v)} \cap \mathcal{M}'$ is the exceptional divisor of the contraction map ψ. This exceptional locus contains an open subset U such that $\psi|_U$ is a $\mathbb{P}^{13}$-bundle over a 10-dimensional base. It degenerates to a 14-dimensional cone over a quartic with the vertex a $\mathbb{P}^9$-bundle as a fiber over a 7-dimensional base.

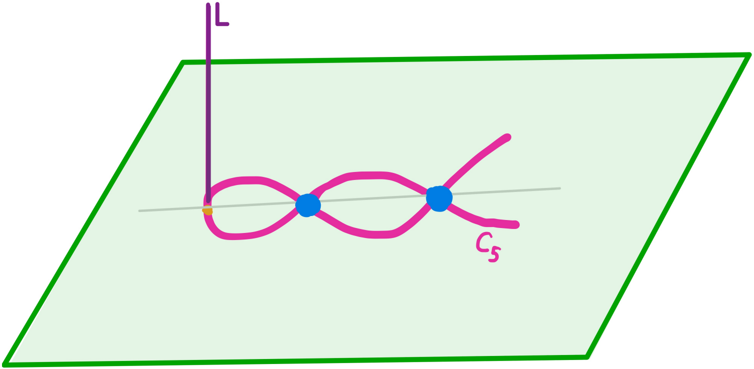
Idea of the proof of Theorem 2

1. Subtle Ext-computations.
2. Technical lemmas.
3. The new component contains stable pairs whose underlying curve is the union of a plane quintic with a line intersecting this quintic, along with two marked points on the quintic.



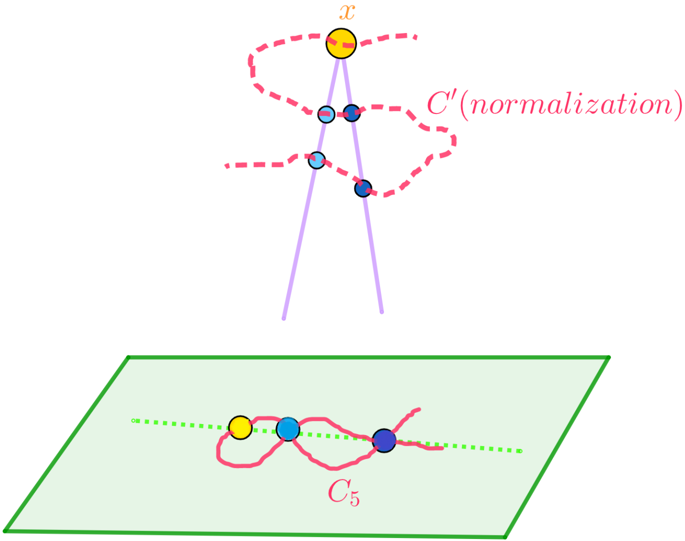
(Idea of the proof of Theorem 2)

4. The stable pairs arise as the degeneration of the ideal sheaf of (2,3)-complete intersection curves $\Longleftrightarrow$ the quintic has two nodes that are colinear with the intersection point with the line, and if the two marked points are the nodes.



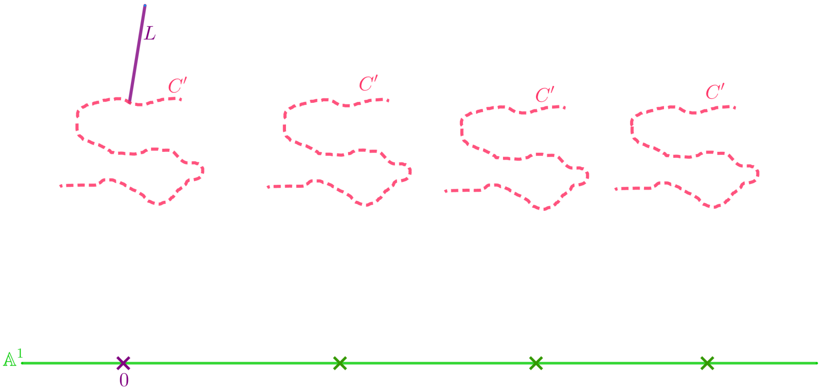
(Idea of the proof of Theorem 2)

4.1. (Partial) Normalization $\rightsquigarrow$ canonical genus four curve C'



(Idea of the proof of Theorem 2)
Degeneration of the normalization

4.2. Construct a family $\mathcal{C} = Bl_0(C' \times \mathbb{A}^1)$ of normalized curves C':

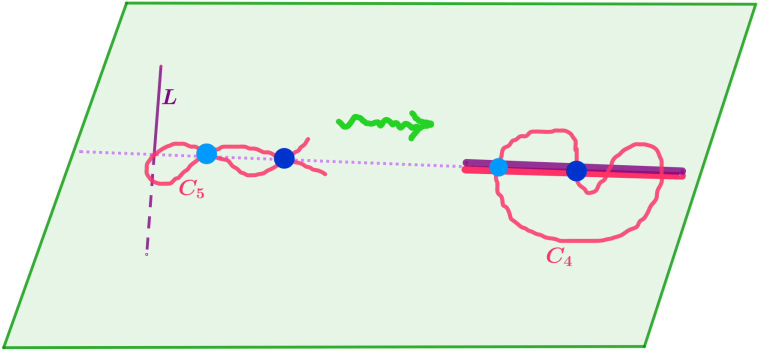


(Idea of the proof of Theorem 2)

4.3. The plane quintic arises as the projection of a (2,3)-complete intersection curve in $\mathbb{P}^3$ from the intersection point with the line.

5. Construct as many objects as possible in the limit of the $\mathbb{P}^{13}$-bundle to recover the 14-dimensional cone in its closure.

5.1. degenerate $C_5 \cup L$ to $C_4 \cup D$, where C_4 a plane quartic and D a thickened line

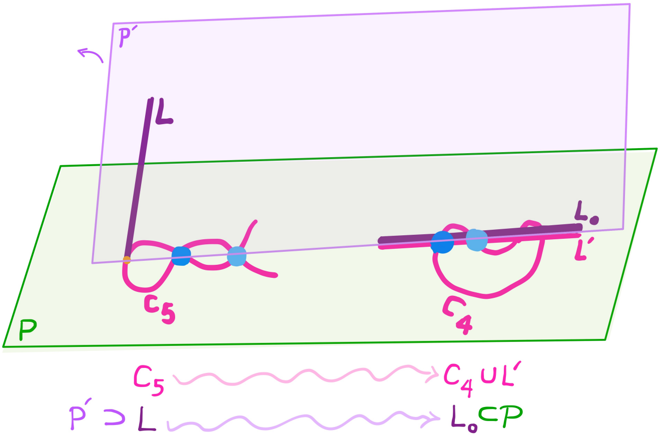


(Idea of the proof of Theorem 2)

5.2. 12 (choice of C_4) + 2 (2 parameters for infinitesimal thickening direction)= 14-dimensional cone.

infinitesimal parameters:

- proportion of the deformations of L and C_5
- deformation of the plane P' (containing L)

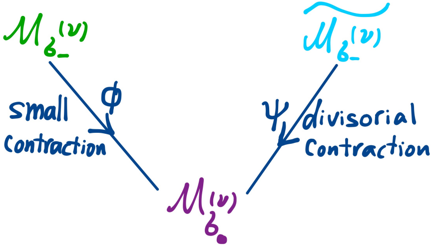


Corollary of Theorem 2

After giving a description of the singular locus of $\mathcal{M}'$, and then using Theorem 2 we will get:

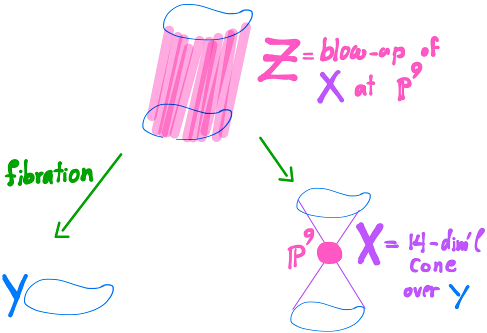
$\rightsquigarrow$ Singular locus of $\mathcal{M}_{\sigma_+}(v)$
= Intersection of the two components of $\mathcal{M}_{\sigma_+}(v)$
= Exceptional locus of ψ

Corollary (of Theorem 2). ψ is a divisorial contraction.



Idea of the proof of relative Picard rank=1

- **The relative Picard rank of ϕ is one:** Non-trivial fibers of ϕ are $\mathbb{P}^1$s which are all numerically equivalent (they occur in a connected family).
- **The relative Picard rank of ψ is one:** Enough to show the fibers have $1-$dimensional N_1 (numerical group of 1-cycles):
  - **$\mathbb{P}^{13}$:** Projective contraction.
  - **14-dim cone:** Extend the method in [Fulger-Lehmann] from a cone with point vertex to the one with the $\mathbb{P}^9$ vertex (using the relation between $N_1(X), N_1(Y), N_1(Z), N_0(Y)$):



Proof of non-$\mathbb{Q}$-factoriality of $\widetilde{\mathcal{M}_{\sigma_-}(v)}$

If it was $\mathbb{Q}$-factorial,

- ψ is a divisorial contraction
- ψ is a of relative Picard rank one
- $\mathcal{M}_{\sigma_0}(v)$ is the image of $\widetilde{\mathcal{M}_{\sigma_-}(v)}$ under ψ

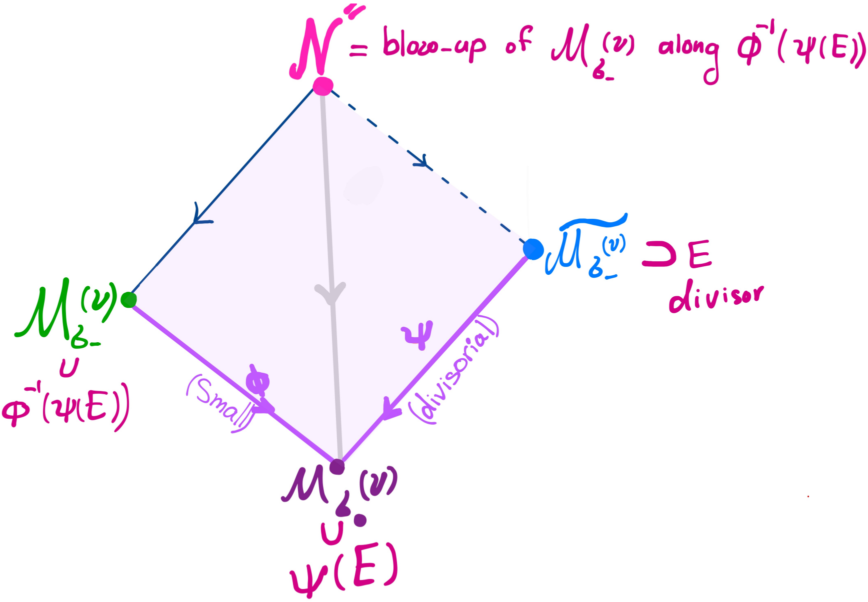
$\Rightarrow$ (1) $\mathcal{M}_{\sigma_0}(v)$ would also be $\mathbb{Q}$-factorial ([Kollár-Mori]).

On the other hand, $\mathcal{M}_{\sigma_0}(v)$ is the image of the $\mathbb{Q}$-factorial variety $\mathcal{M}_{\sigma_-}(v)$ under a small contraction,

$\Rightarrow$ (2) $\mathcal{M}_{\sigma_0}(v)$ cannot be $\mathbb{Q}-$factorial.

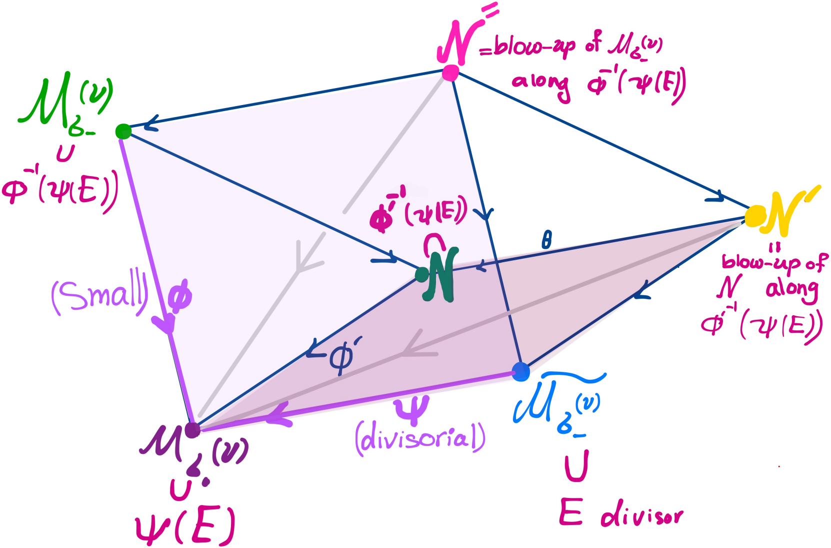
(1), (2) $\Rightarrow$ contradiction.

Birational relationship between $\mathcal{M}_{\sigma_-}(v)$ and $\widetilde{\mathcal{M}_{\sigma_-}}(v)$



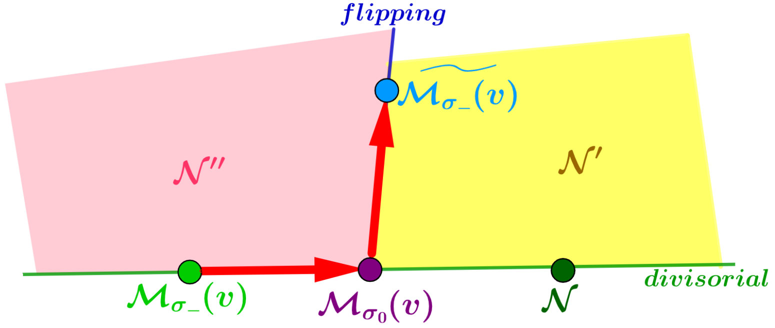
(Birational relationship between $\mathcal{M}_{\sigma_-}(v)$ and $\widetilde{\mathcal{M}_{\sigma_-}(v)}$)

$\mathcal{N}$: Flip of $\mathcal{M}_{\sigma_-}(v)$



Movable cone of the blow-up of $\mathcal{M}_{\sigma_-}(v)$

$\mathcal{N}$: Flip of $\mathcal{M}_{\sigma_-}(v)$
$\mathcal{N}'$: Blow-up of $\mathcal{N}$
$\mathcal{N}''$: Blow-up of $\mathcal{M}_{\sigma_-}(v)$



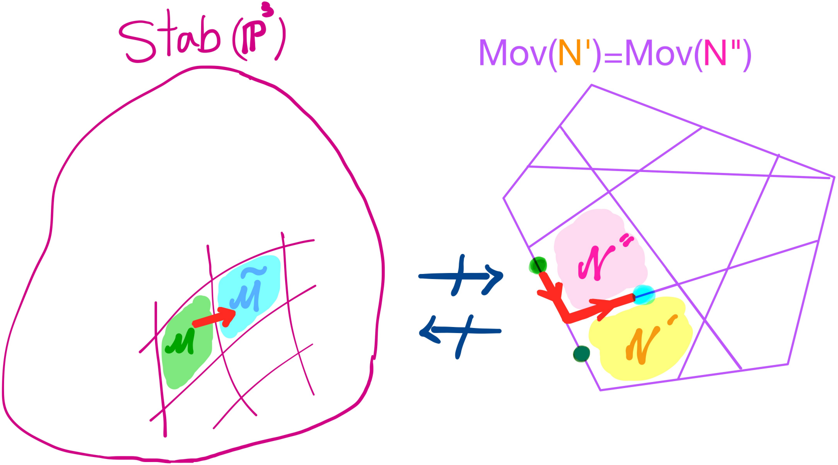


Figure 14: Correspondence fails

Thank you for your attention!