Degenerating tangent curves

(joint with Lawrence Barrott)

Aim: to be able to explain to you the following picture:

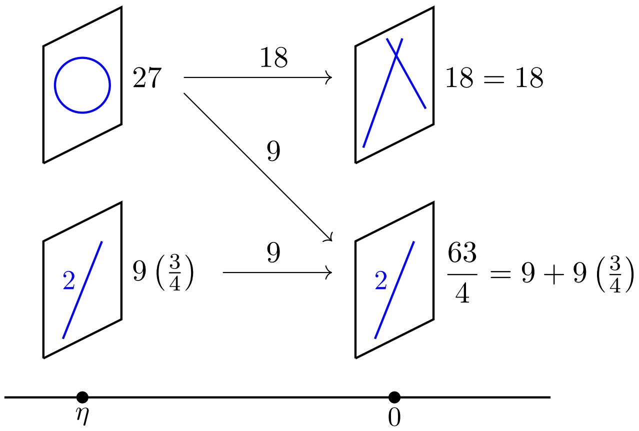

FIGURE 1. Degree 2 degeneration. Total invariant is: $27 + 9(3/4) = 18 + 63/4 = 135/4$.

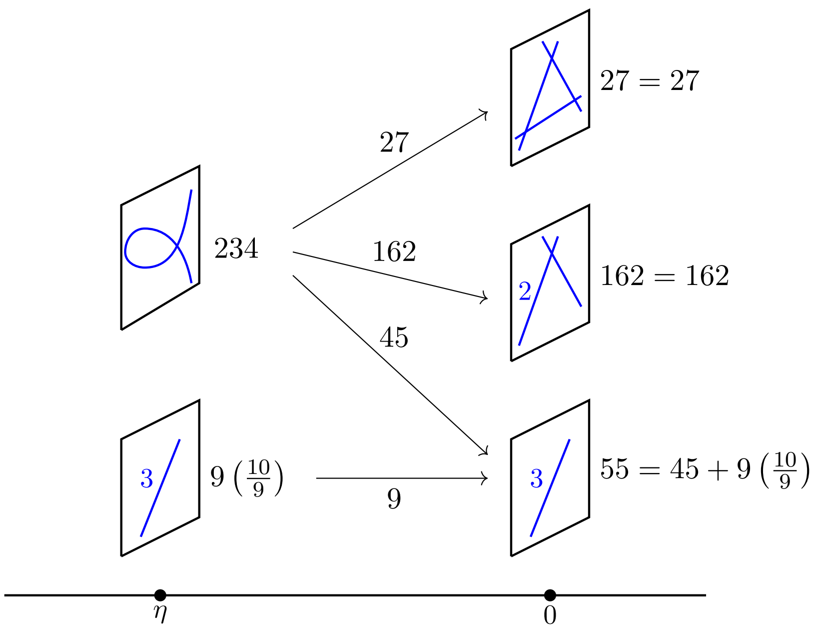

FIGURE 2. Degree 3 degeneration. Total invariant is: $234+9(10/9) = 27+162+55 = 244$.

Picture of a theorem which is classical in character.

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Proof hinges on modern techniques: log Gromov–Witten theory, log deformation theory, localisation...

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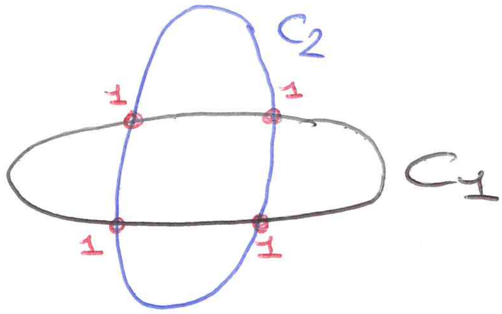
1) Enumerative geometry with tangency conditions.

Take 2 smooth conics:

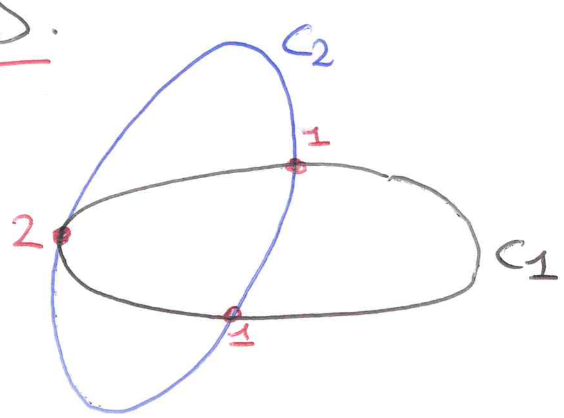
$$\boxed{C_1, C_2 \subseteq \mathbb{P}^2}$$

Bézout: $\#(C_1 \cap C_2) = 4$, counting with multiplicity.

- Generically, all multiplicities are 1:



- But for certain choices of conics, have fewer points with larger mults:



- In this case, say: C_1 is tangent to C_2.

- We're going to cook up an enumerative geometry problem involving conics and tangency.

- Setup: fix 5 smooth conics:

$$\boxed{C_1, \dots, C_5 \subseteq \mathbb{P}^2}$$

in general position.

- consider moduli space of all smooth conics. Is dense open:

$$\boxed{U \underset{\text{open}}{\subseteq} \mathbb{P}^5 \cong \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(2)).}$$

- For $i \in \{1, \dots, 5\}$, general element of U intersects C_i transversely.

$\downarrow$

Look instead at special locus

$$\boxed{V_i \subseteq U}$$

consisting of smooth conics $C \in U$ which are tangent to C_i.

- Fact (Plausible): $V_i \subseteq U$ a hypersurface.

- Now consider conics $C \in U$ tangent to $C_1, \dots, C_5$ simultaneously:

$$\boxed{\bigcap_{i=1}^{5} V_i \subseteq U.}$$

Expect to get finite collection.

$\downarrow$

Question: how many?

⋮

Answer: 3264.

(Number of smooth conics tangent to 5 fixed conics.)

- Proof: $\deg(\overline{V}_i) = 6$.

$$\Downarrow$$

Intersection in $\mathbb{P}^5$ gives:

$6^5 = 7776$ Points.

$$\downarrow$$

- Wrong answer! (Steiner)

- Why? Intersection in $\mathbb{P}^5$ includes entire locus of double lines (2D locus in $\mathbb{P}^5$). (Contained in $\mathbb{P}^5 / (4.)$)

$$\downarrow$$

Need to remove this contribution; excess intersection formula:

$$\boxed{7776 - 4512 = 3264.}$$

- Let $j_i : V_i \hookrightarrow \mathbb{P}^5$ for $i \in \{1, .., 5\}$

$\Rightarrow (j_i)_*[V_i] = 6 \cdot H \in A^1(\mathbb{P}^5)$.

$\Rightarrow$ $\boxed{\prod_{i=1}^5 (j_i)_*[V_i] = 6^5 [pt] \in A^5(\mathbb{P}^5)}$.

- But refined intersection product [Fulton-MacPherson] allows us to express the intersection class as a pushforward from the physical intersection:

- I.e. $\exists$ $\boxed{\gamma \in A^*\left(\bigcap_{i=1}^5 V_i\right)}$ with:

$\boxed{j_*(\gamma) = 6^5 [pt]}$.

This γ carries more information!

- $\bigcap_{i=1}^5 V_i$ has many components.

$\Rightarrow$ get contributions.

— This is the flavour of question we are interested in:

$\boxed{\text{Fix collection of divisors, and count curves with fixed tangency orders to these divisors.}}$

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2) Log Gromov-Witten theory

— Want to do enumerative geometry with tangencies, from point of view of Gromov-Witten theory.

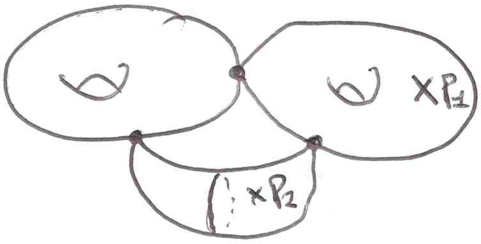
- Fix ambient space X (smooth + projective).

Space of stable maps:

$$\overline{M}_{g,n}(X,\beta) = \left\{ \begin{array}{c} C \xrightarrow{f} X \\ p_1,\dots,p_n \in C \end{array} \middle| \begin{array}{l} C \text{ nodal projective} \\ p_i \text{ distinct smooth pts} \\ f \text{ stable} \end{array} \right\}$$

genus(C)

$f_*[C] \in H_2(X)$ "degree" of f.



$\xrightarrow{f} X$

- Compactification of space of parametrised smooth curves in X:

$C \xhookrightarrow{i} X$.

- Note: f not an embedding in general.

- This is our "moduli space of curves in X"; analogue of $\mathcal{M}$ from before.

- As before, get enumerative invariants by imposing some conditions to cut down moduli space to a finite collection of points.

$\downarrow$

- E.g: incidence conditions: curve intersects a subvariety $\boxed{Z \subseteq X}$.

In GW theory, impose these using evaluation maps:

$$\boxed{\left.\begin{aligned} ev_i : \overline{M}_{g,n}(X,\beta) &\longrightarrow X \\ [C, f, p_1, \ldots, p_n] &\longmapsto f(p_i) \end{aligned}\right\}}$$

- Then $ev_i^{-1}(Z)$ = locus of stable maps sending p_i to Z.

- Imposing one such condition for each p_i, get an intersection:

$$\int_{[\overline{M}_{g,n}(X,\beta)]} \prod_{i=1}^{n} ev_i^*[Z_i]$$

This is called a Gromov-Witten invariant

- $\boxed{\text{Enumerative geometry} = \text{Intersection theory on moduli spaces.}}$

(Need: compact/proper, (virtually) smooth.)

- Sometimes GW invariants coincide with "classical" counts, sometimes not.

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Theory powerful: recursive structure of moduli space makes possible to prove deep results.

↓

> E.g: [Kontsevich, '90s]:
> # deg. d rational plane curves through $3d-1$ general points.

- Now want to incorporate tangencies into this story.

- Fix hypersurface $\boxed{D \subseteq X}$.

- Recall we have marked points:

$$p_1, \ldots, p_n \in C \xrightarrow{f} X.$$

we will impose tangency of the map to D at the p_i.

- Fix a partition:

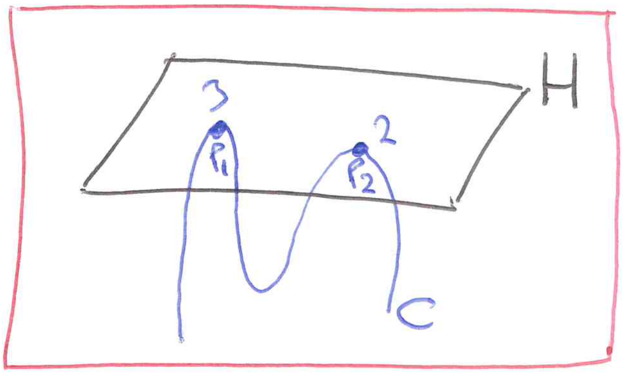
$$\boxed{\alpha = (\alpha_1, \ldots, \alpha_n) \vdash D \cdot \beta}$$

want to consider maps with tangency order α_i to D at p_i.

E.g.: $X = \mathbb{P}^3$, $D = H$, $\beta = 5 \cdot L$.

$$\boxed{\alpha = (3, 2)}$$

P_1 P_2



— Want to define a corresponding moduli space

$$\boxed{\overline{M}_{g,\alpha}(X|D, \beta)}$$

of relative stable maps.

$\downarrow$

— Desiderata:

1. Compact
2. (Virtually) smooth.

— How to define this?

— When $f^{-1}(D)$ consists of isolated points, clear what we need to do.

$\downarrow$

Require that every point in $f^{-1}D$ is marked, with contact order as specified by α.

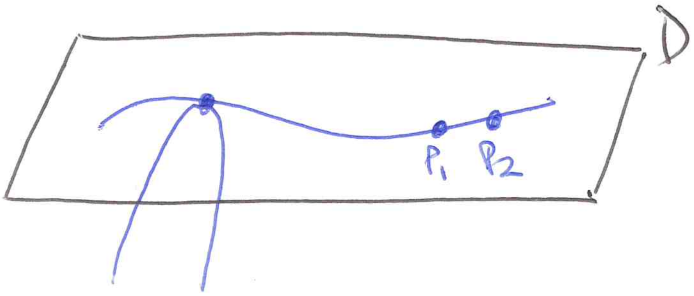
$\downarrow$

Get a ~~closed~~ subspace:

$$\boxed{M_{g,\alpha}(X|D,\beta) \subseteq \overline{M}_{g,n}(X,\beta).}$$

— Problem: this is not compact.

In the limit can have whole components of C mapping into D:



- Can no longer measure tangency order at internal markings.

↓

- Need a way to keep track of these even as curve falls inside D.

- Several different solutions to this problem.

↓

- Modern approach: log stable maps.

- Idea: attach some extra structure to C and X, to allow us to make sense of tangency, even as curve falls into D.

Def^n (unorthodox: [Borne–Vistoli]):

Let X be a scheme. A log structure on X consists of the following data:

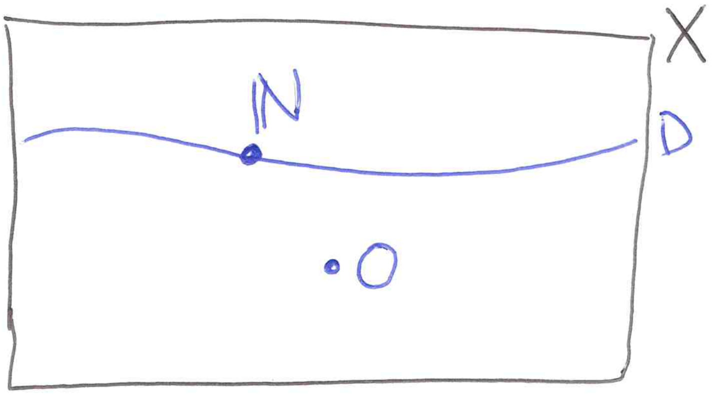
(i) a constructible sheaf of monoids $\boxed{\overline{M}_X}$ (the ghost sheaf). ← "discrete" part

(ii) a rule for associating, to every section of $\overline{M}_X$, a line bundle–section pair: ← "continuous" part

$$\boxed{\varphi \in \overline{M}_X \rightsquigarrow (\mathcal{O}_X(\varphi), s_\varphi).}$$

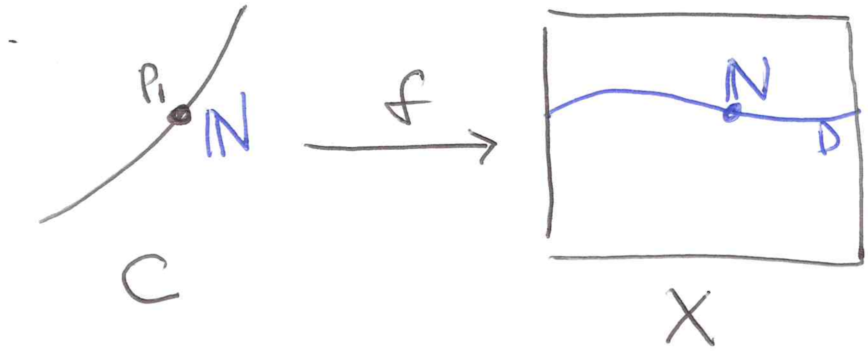
E.g.: $D \subseteq X$ smooth divisor

$\rightsquigarrow$ divisorial log structure on X:



$$\mathbb{N} \ni 1 \longmapsto (\mathcal{O}_X(D), s_D).$$

- Anatomy of log stable map:



- To enhance f to a map of log schemes, need:

$$f^{-1}\overline{M}_X \longrightarrow \overline{M}_C.$$

Over $p_1 \in C$ this gives:

$$f^{-1}\overline{M}_X|_{p_1} = \mathbb{N} \longrightarrow \mathbb{N} = \overline{M}_C|_{p_1}$$

which we think of as the tangency order.

- Compatibility with line-bundle-section pairs ensures this agrees with the "true" tangency when C is not mapped inside D.

- But this data well-defined everywhere!

- Upshot: get proper, virtually smooth moduli space

  $$\boxed{\overline{M}^{\log}_{g,\alpha}(X|D,\beta).}$$

  of log stable maps.

- Doing intersection theory on this space produces enumerative counts, called log Gromov-Witten invariants.

- History:
  - Siebert lecture (early '00s).
  - Abramovich - Chen - Gross - Siebert.
  - Expanded in multiple directions:
    - tropical curve counting;
    - degeneration formulae;
    - mirror symmetry.
    - etc...

- Log structures have opened the way for new techniques to enter the subject.

$\downarrow$

- $\overline{M}_X$ gives stratification of X, analogous to stratification of toric variety into torus orbits.
- Can think of a log scheme as having "local structure of a toric variety"

$\downarrow$

- To a log scheme, associate a combinatorial object called the tropicalisation, analogous to the fan.

- Use to import techniques from toric geometry:

- $\boxed{\text{Subdivisions of } \mathrm{Trop}\, X \longleftrightarrow \text{birational modifications of } X}$
- $\boxed{\text{PL fns on } \mathrm{Trop}\, X. \longleftrightarrow \text{Cartier divisors on } X.}$

- In log GW theory, moduli spaces carry tautological log structure

$$\Downarrow$$

- Can use "toric" techniques to probe geometry of moduli spaces.
- Interplay with tropical geometry key.

3) Tangent curves to degenerating hypersurfaces

(Joint w/ Lawrence Barrott.)

— Setup: fix $\boxed{E \subseteq \mathbb{P}^2 \text{ smooth cubic}}$

— Study genus 0, degree d curves/ stable maps in $\mathbb{P}^2$ meeting E in a single point.

(Hence with maximal tangency order $3d$.)

— Expect finitely many. Can try to count them. Can ask for:

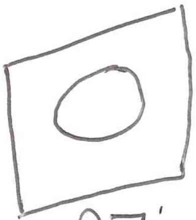
(i) Classical count (honest-to-God, embedded integral rational curves).

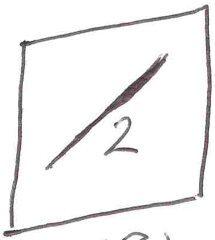
(ii) log GW invariant.

- These are <u>not the same</u>! we'll see why in a minute.

- <u>E.g.</u> $d=1$.
  - Classically, we know E has <u>9 flex lines</u>.
  - Log GW invariant is also 9: space of log stable maps consists of <u>9</u> isolated points.

- <u>E.g.</u> $d=2$.
  - classical count: <u>27 smooth conics</u>.
  - Log stable maps has:
    (i) 27 isolated pts corresponding to smooth conics.
    (ii) 9 1D components, corresponding to <u>double covers of flex lines</u>.





$$27 + 9\left(\tfrac{3}{4}\right) = \frac{135}{4}$$

log GW invariant

excess intersection calculation [GPS].

— General Phenomenon here: GW invariant made up of contributions of different components.

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Some represent classical curves, others represent degenerate objects.

— GW invariants easier to compute than classical counts ($\exists$ methods).

↳ $\boxed{\exists \text{ closed formula for invariants of } (\mathbb{P}^2, E) \text{ [Gathmann, Takahashi]}}$

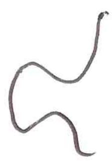
— The game: can we unravel the individual contributions?

(can we deduce classical counts from GW counts?)

$\downarrow$

[Nobuyoshi Takahashi,
Gross–Pandharipande–Siebert,
Choi–van-Garrel–Katz–Takahashi,
Bousseau, Gräfnitz, ...]

— Our question: degenerate E to union of co-ordinate lines in $\mathbb{P}^2$:

$E =$  $\rightsquigarrow$  $= \Delta \subseteq \mathbb{P}^2$

- Tangent curves will degenerate along with E.

$\boxed{\text{Q: What do they limit to?}}$

- Notice: given degenerating family of tangent curves:

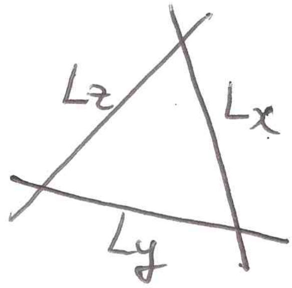
$$C_t \rightsquigarrow C_0$$

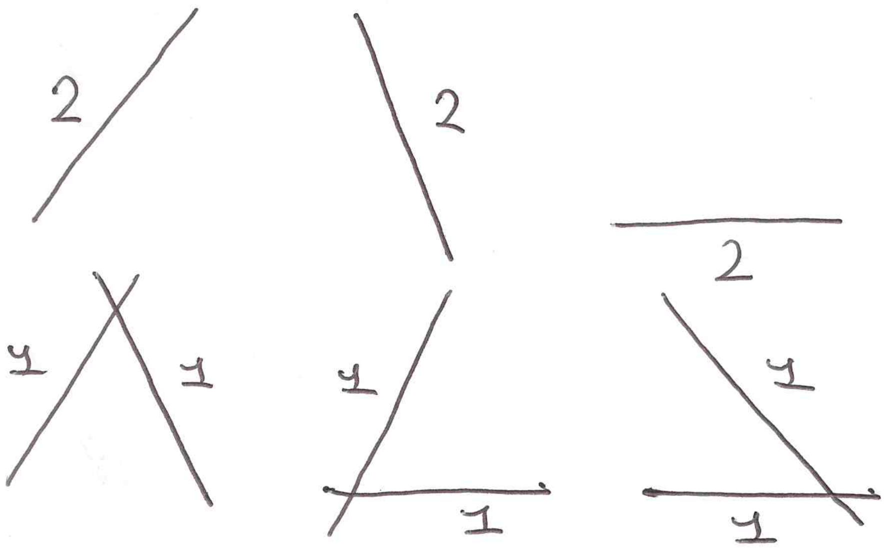
(tangent to $E=E_t$)

we must have $\boxed{C_0 \subseteq \Delta}$ ⊛.

- (Otherwise, would intersect Δ in ≥ 2 points.)

– So $\boxed{C_0 \subseteq \Delta}$. Possibilities given by splitting of degree.

– E.g.: $d = 2$. $\Delta =$ 



– Q: Of 27 smooth conics in general fibre, how many limit to (2) and how many to (1,1)?

- Solution: log GW theory.
- Have family of moduli spaces:

$$\begin{array}{c} \overline{M}^{\log}_{0,(3d)}(\mathbb{P}^2 | E_t, d) \\ \downarrow \pi \\ \mathbb{A}^1_t \end{array}$$

- General fibres all the same.
- Central fibre is maps to Δ:

$$\overline{M}_{0,1}(\Delta, d).$$

Breaks into components, according to degree splitting.
- Typically very big

[redacted]

- We use log deformation theory to construct a class on central fibre whose integral equals the log GW invariant of general fibre.

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- Refines log GW invariant, as sum over components of the central fibre.

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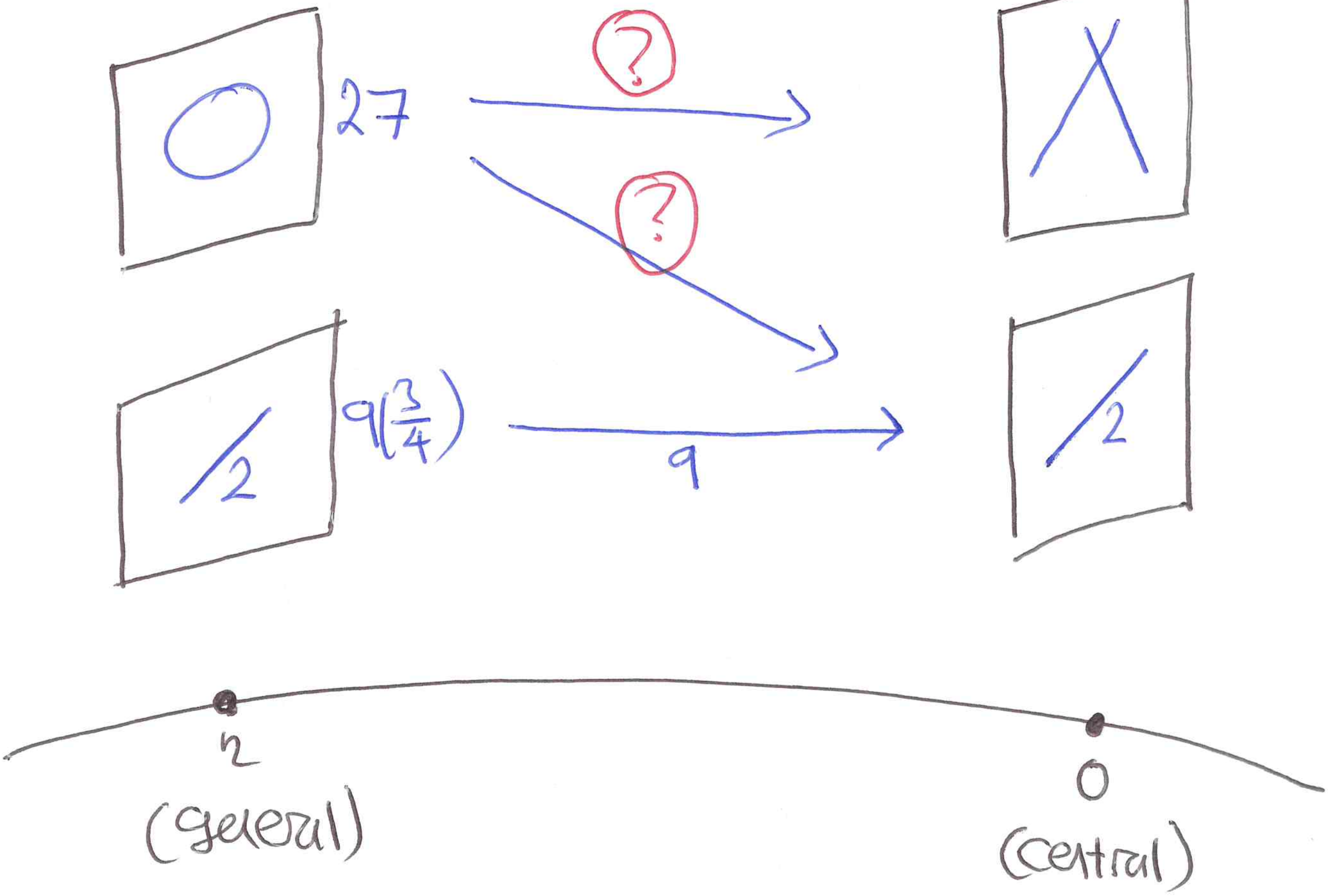
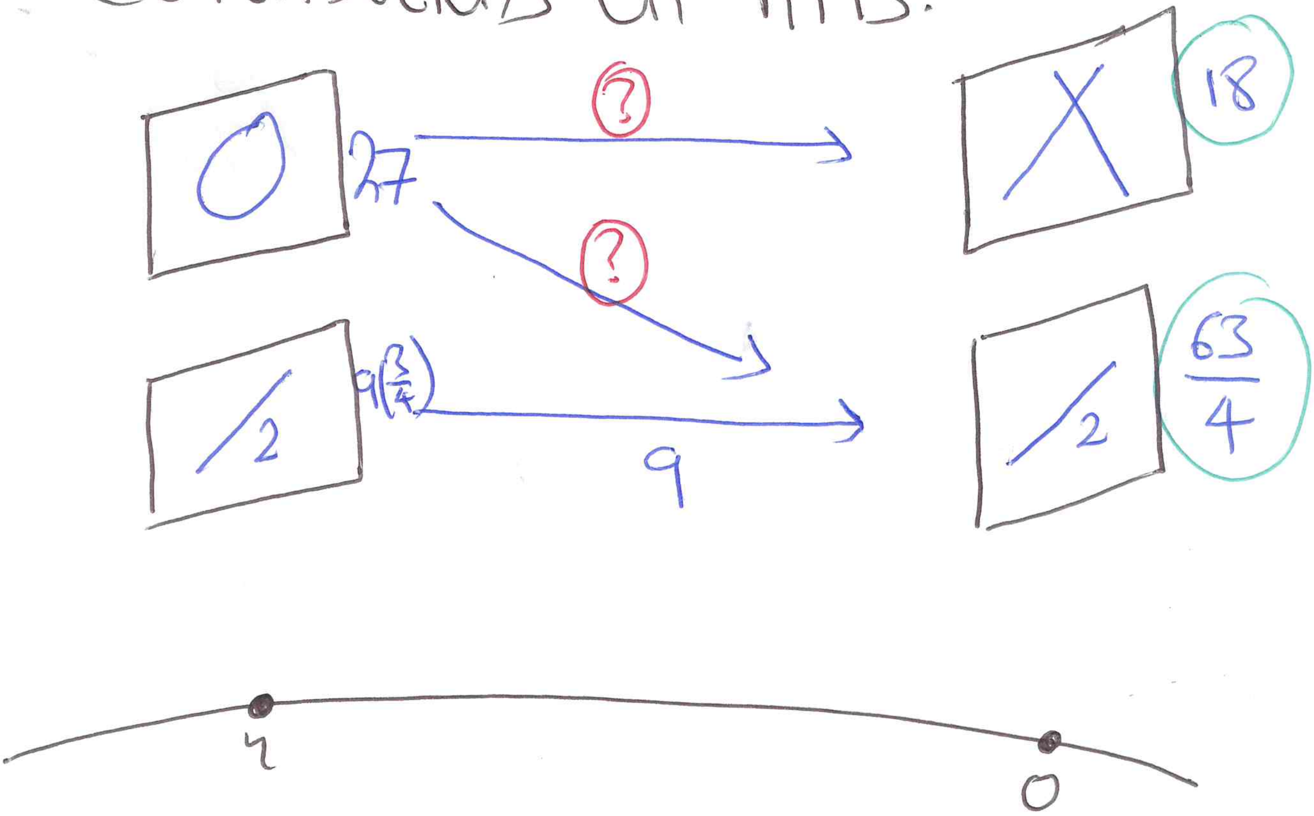
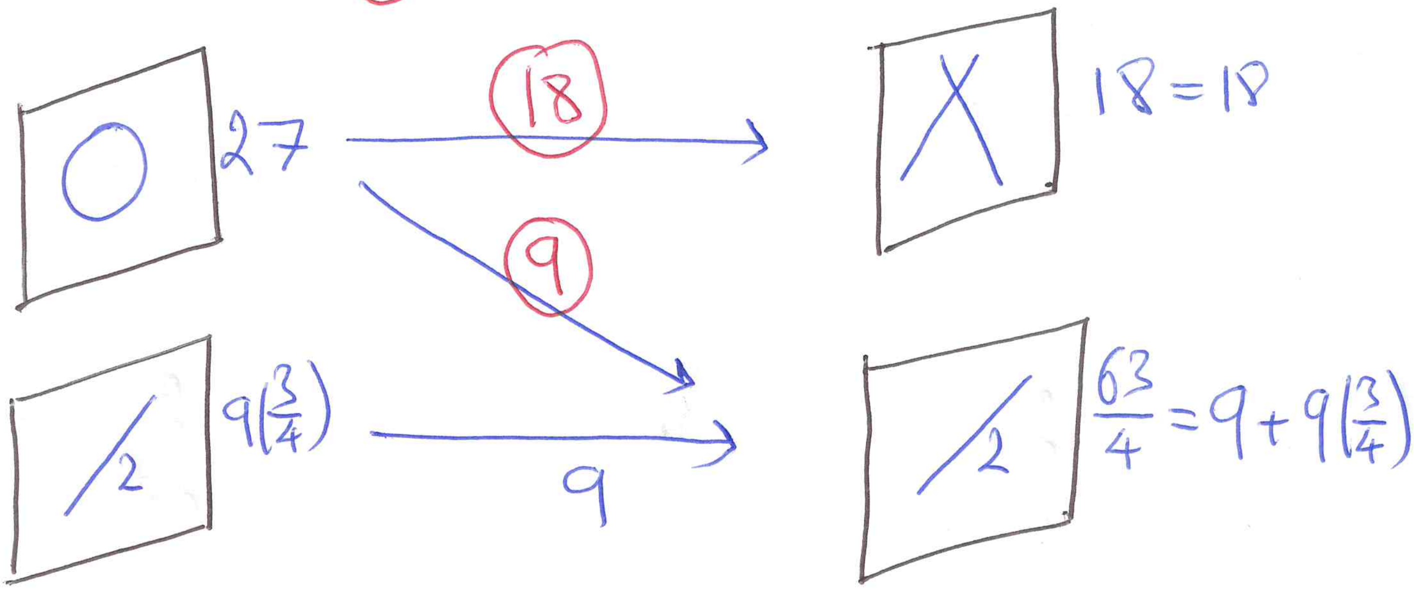
- We compute these contributions (hard: localisation and tropical techniques).



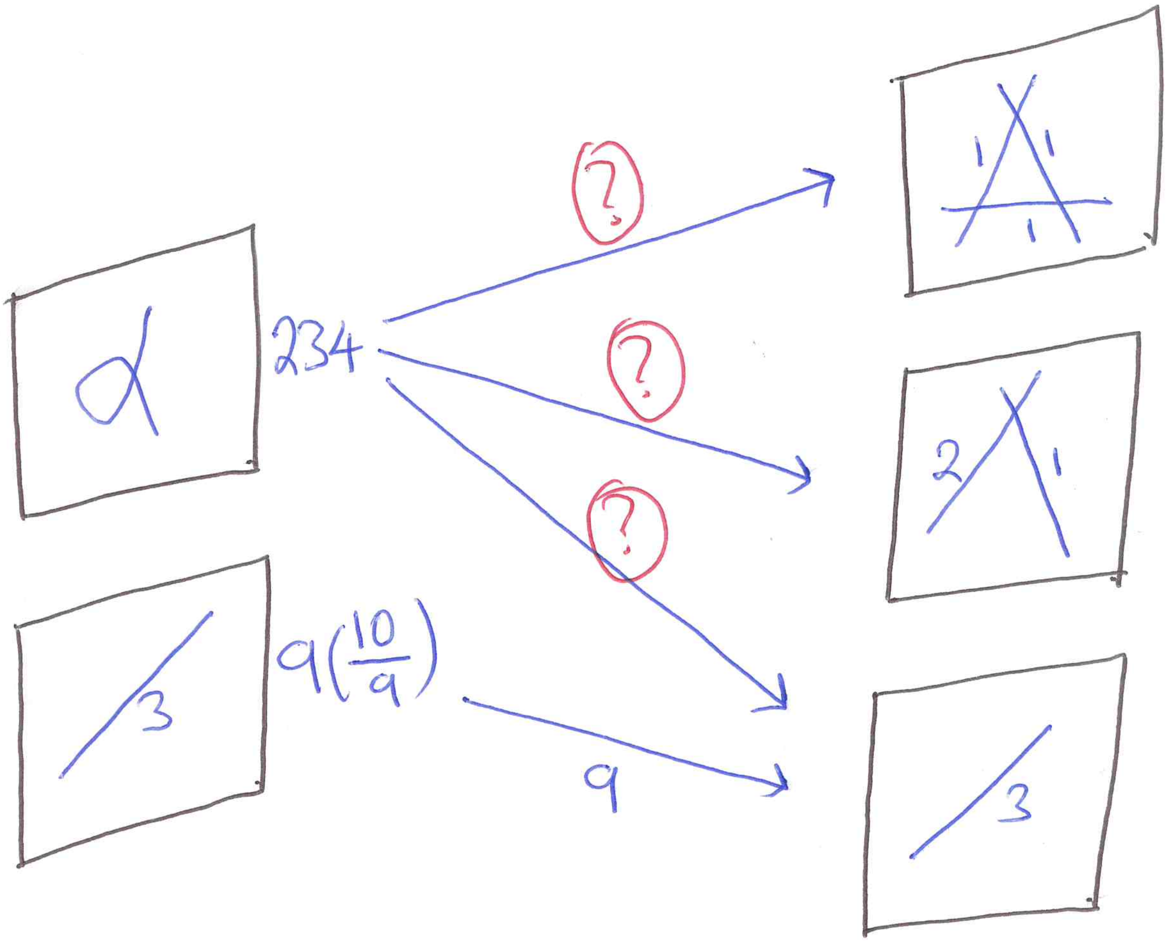
- Point is: we now know contributions on RHS.



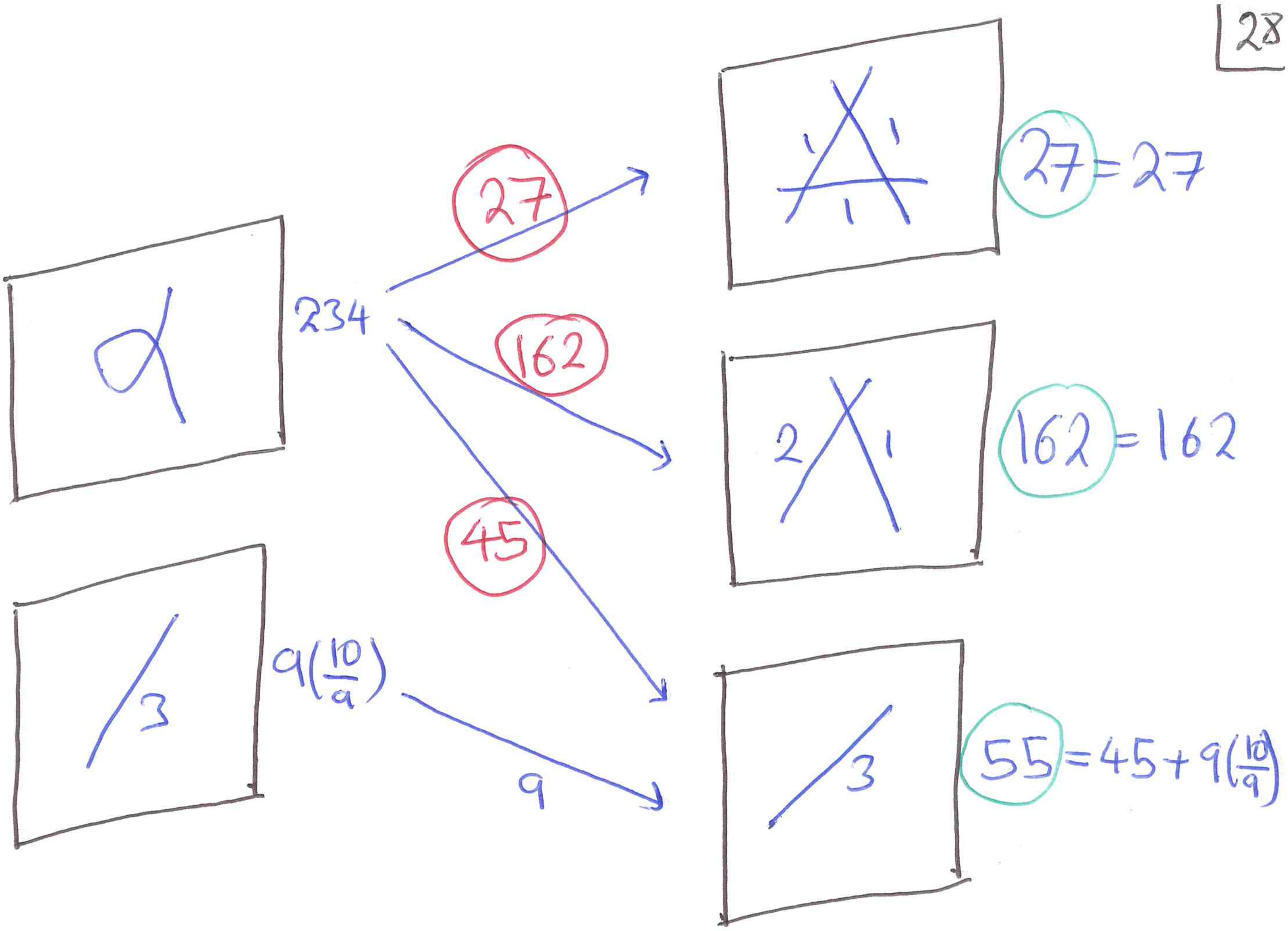
- So can work backwards to deduce ?:



- Similarly for $d=3$:



Again, calculate RHS contributions and then unravel:



Thank you!