"On the Liftability of the Automorphism Group of smooth Hypersurfaces of the Projective Space" / $\mathbb{C}$

[joint work with Victor GONZALEZ-AGUILERA and Alvaro LIENDO]

§1. Notation, History and Motivation

Let us consider $F \in \mathbb{C}[x_0, \dots, x_{n+1}]$ homogeneous $\neq 0$ of degree d and let

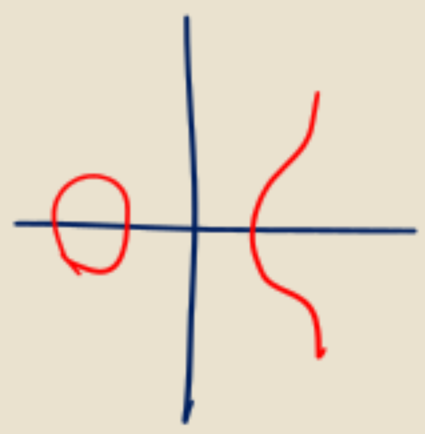
$$X := \{x \in \mathbb{P}^{n+1} \text{ st } F(x) = 0\} \subseteq \mathbb{P}^{n+1}$$

be a SMOOTH hypersurface of $\begin{cases} \dim(X) = n \geq 1 \\ \deg(X) = d \geq 3 \end{cases}$

Examples (that will appear later):

a) ELLIPTIC CURVES: Let $a, b \in \mathbb{C}$

$$C := \{[x, y, z] \in \mathbb{P}^2 \text{ st } y^2 z = x^3 + axz^2 + bz^3\} \subseteq \mathbb{P}^2$$



is smooth $\Longleftrightarrow \Delta = 4a^3 + 27b^2 \neq 0 \rightsquigarrow C$ is an elliptic curve.

b) QUARTIC K3 SURFACES: $S \subseteq \mathbb{P}^3$ smooth surface of degree 4 ($\Rightarrow \pi_1(S) \simeq \{1\}$ and $\omega_S \cong \mathcal{O}_S$, ie, S is K3).

(Construction) Consider 4 bilinear equations of the form

$$H_K = \{\sum_{i,j=0}^{3} a_{ij}^{K} x_i y_j = 0\} \subseteq \mathbb{P}^3_x \times \mathbb{P}^3_y \quad (K = 1, \dots, 4)$$

and let $S := H_1 \cap H_2 \cap H_3 \cap H_4 \subseteq \mathbb{P}^3 \times \mathbb{P}^3$.

What is $pr_1(S) \subseteq \mathbb{P}^3_x$?

Note that $p \in \mathbb{P}^3_x$ belongs to $pr_1(S)$ if and only if $B(p)\, y = 0$ has a solution $y_0 \neq 0$ $\iff$ $\det B(p) = 0$

↳ $B = (b_{kj})$ with $b_{kj} = \sum_i a^k_{ij} x_j$ linear form

⇝ $pr_1(S) = \{ x \in \mathbb{P}^3 \text{ st } \det B(x) = 0 \} =: S_1$ ← degree 4 surface

Similarly: $pr_2(S) = \{ y \in \mathbb{P}^3 \text{ st } \det C(y) = 0 \} =: S_2$

↳ $C = (c_{ki}) = \left(\sum_j a^k_{ij} y_j \right)$

FACT: If the coefficients a^k_{ij} are GENERAL then:

i) $S \subseteq \mathbb{P}^3 \times \mathbb{P}^3$ is smooth ✓

ii) $S \cong S_1 \subseteq \mathbb{P}^3$ and $S \cong S_2 \subseteq \mathbb{P}^3$ via the projections ✓

c) FERMAT HYPERSURFACE:

$$X_F = \{ x \in \mathbb{P}^{n+1} \text{ st } x_0^d + x_1^d + \dots + x_{n+1}^d = 0 \} \quad \text{smooth ✓}$$

d) KLEIN HYPERSURFACE:

$$X_K = \{ x \in \mathbb{P}^{n+1} \text{ st } x_0^{d-1} x_1 + x_1^{d-1} x_2 + \dots + x_n^{d-1} x_{n+1} + x_{n+1}^{d-1} x_0 = 0 \}$$

smooth (since $d \geq 3$) ✓

Question: Given two smooth hypersurfaces $X_1, X_2 \subseteq \mathbb{P}^{n+1}$, when $X_1 \cong X_2$ as abstract/embedded varieties?

If $X_1 = X_2 = X$ one should look at

$$\mathrm{Aut}(X) := \{ \varphi : X \xrightarrow{\sim} X \text{ isomorphism (as abstract variety)} \}$$

GROUP OF (REGULAR) AUTOMORPHISMS.

Examples:

a) $\mathrm{Aut}(\mathbb{P}^{n+1}) \cong PGL_{n+2}(\mathbb{C})$

b) Given $X \subseteq \mathbb{P}^{n+1}$ hypersurface, we define

$$\mathrm{Lin}(X) := \{ \varphi \in \mathrm{Aut}(X) \text{ st } \exists\, \Phi \in PGL_{n+2}(\mathbb{C}) \text{ st } \varphi = \Phi|_X \}$$

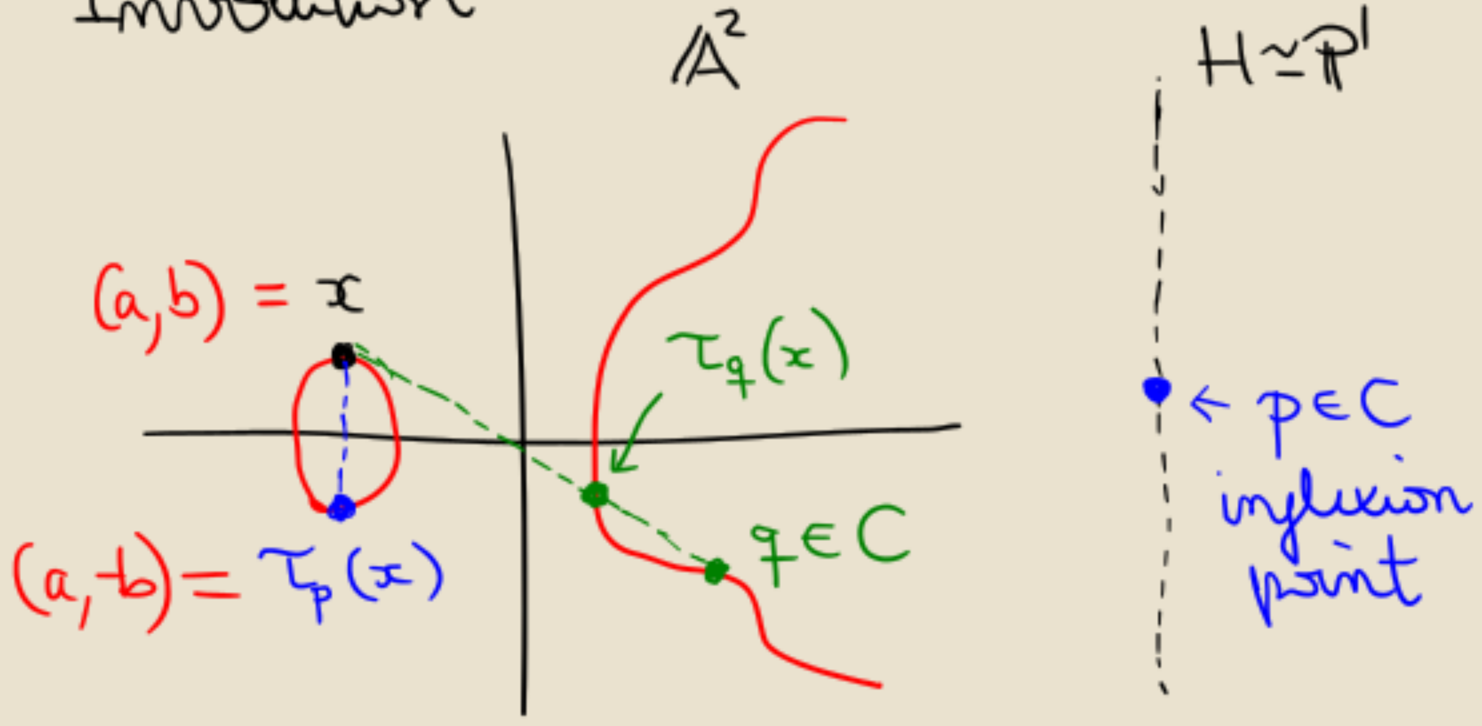
the group of LINEAR automorphisms.

(e.g. permutation matrices act on $X_F = \{ x_0^d + \ldots + x_{n+1}^d = 0 \}$)

c) ELLIPTIC CURVES: Fix $p_0 \in C$ and let

$\tau_{p_0} : C \longrightarrow C$, $x \longmapsto$ 3rd intersection point of $\langle p_0, x \rangle$ with C

Involution



$\tau_{p_0} \in \mathrm{Lin}(C)$ if and only if p_0 is an inflexion point

$\Longrightarrow \mathrm{Lin}(C) \subsetneq \mathrm{Aut}(C)$!

d) QUARTIC K3 SURFACES: With the same notation as before, we have

$$\underbrace{S \cong \{ \det B = 0 \} \subseteq \mathbb{P}^3}_{= S_1} \quad \text{and} \quad S \cong \underbrace{\{ \det C = 0 \}}_{= S_2} \subseteq \mathbb{P}^3$$

[Segre ≈ 44]

The Cramer's rule from Linear Algebra allows us to construct an isomorphism $\varphi : S_1 \xrightarrow{\sim} S_2$ given by degree 3 polynomials (3×3 minors!) $\underset{(\ldots)}{\rightsquigarrow}$ $\mathrm{Lin}(S_1) \subsetneq \mathrm{Aut}(S_1)$

Thm (Matsumura–Monsky 1964): $X \subseteq \mathbb{P}^{n+1}$ smooth hypersurface of dim $n \geq 1$ and degree $d \geq 3$. Then:

1° $\mathrm{Lin}(X)$ is a finite group (cf. Jordan 1880)

2° If X is general (in its moduli space) $\Rightarrow \mathrm{Aut}(X) = \{ \mathrm{Id}_X \}$.

3° If $(n,d) \notin \{ (1,3), (2,4) \} \Rightarrow \mathrm{Lin}(X) = \mathrm{Aut}(X)$.

Even better: By means of "classical" results (Noether, Lefschetz, Matsusaka, Mumford, ...) together with recent ones (Oguiso 2016, Shimada – Shioda 2017) we have:

Thm: Let $X_1, X_2 \subseteq \mathbb{P}^{n+1}$ be smooth hypersurfaces of degrees $d_1, d_2 \geq 3$ respectively. Suppose that $\varphi : X_1 \xrightarrow{\sim} X_2$ is an isomorphism, then $\exists\, \Phi \in PGL_{n+2}(\mathbb{C})$ st $\varphi = \Phi|_{X_1}$ except (maybe) in the following cases:

1° $n = 1$ and $d_1 = d_2 = 3$ (elliptic curves)

2° $n = 2$ and $d_1 = d_2 = 4$ (quartic K3 surfaces).

§2. Recent works: Let $n, d \in \mathbb{N}^{\geq 1}$ be fixed.

Recall that $\mathcal{H}_n(d) := \mathbb{P}H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d)) \cong \mathbb{P}^N$ ← $N+1 = \binom{n+d+1}{d}$

parametrizes hypersurfaces $X \subseteq \mathbb{P}^{n+1}$ of degree d, and that

$$\mathcal{U}_n(d) := \{ \text{smooth hypersurfaces } X \subseteq \mathbb{P}^{n+1} \text{ of deg } d \} \subseteq \mathcal{H}_n(d)$$

is a Zariski open subset st $\mathcal{H}_n(d) \setminus \mathcal{U}_n(d) = \mathcal{D} = \{\Delta = 0\}$ is a divisor. (Δ: "discriminant")

Assumption: $n \geq 1$, $d \geq 3$ and $(n,d) \notin \{(1,3), (2,4)\}$

↳ In particular, for every $X \in \mathcal{U}_n(d)$ we have that $Aut(X) \subseteq PGL_{n+2}(\mathbb{C})$ is finite.

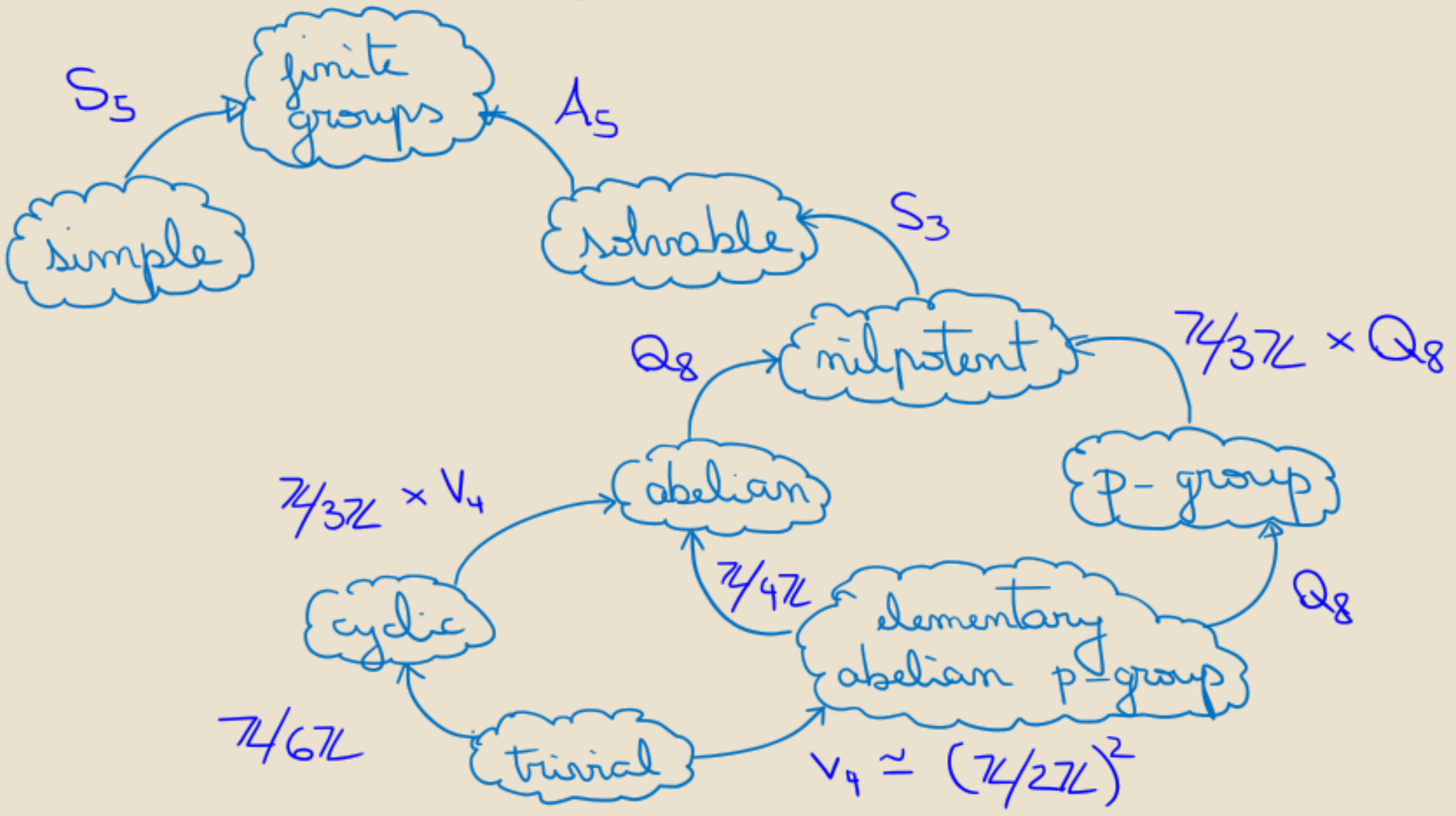
Ehresmann (1951): All the $X \in \mathcal{U}_n(d)$ are diffeomorphic.
In part, $H^n(X, \mathbb{Z}) \cong \mathbb{Z}^{\oplus b_n}$ are constant ← $b_n = \frac{(d-1)^{n+2} + (-1)^n (d-1)}{d}$

Moreover, $Aut(X) \curvearrowright H^n(X, \mathbb{Z})$ faithfully (def. theory)

⇝ Minkowski (1887) $\exists\, C(n,d)$ st $|Aut(X)| \leq C(n,d)$ for all $X \in \mathcal{U}_n(d)$

Main Question: Which finite groups G are ADMISSIBLE in $\mathcal{U}_n(d)$ (i.e., $\exists\, X_d^n \subseteq \mathbb{P}^{n+1}$ smooth st $G \subseteq Aut(X)$) ?

Hierarchy of Finite Groups:



Some results:

✓ Dolgachev & Iskovskikh (2009): $S \subseteq \mathbb{P}^3$ smooth cubic surface $\rightsquigarrow$ Det. all admissible G (11 non-isomorphic of max. order).

✓ González-Aguilera & Liendo (2011, 2013):
Let $q = p^\alpha$ for $\alpha \in \mathbb{N}^{\geq 1}$ and p prime. Assume that q is relatively prime to d and $d-1$. Then:

$$G \simeq \mathbb{Z}/q\mathbb{Z} \text{ admissible in } \mathcal{U}_n(d) \iff \exists\, \ell \in \{1, \dots, n+2\} \text{ st. } (1-d)^\ell \equiv 1 \pmod q.$$

✓ Pambianco (2014) and Harui (2019): $C \subseteq \mathbb{P}^2$ smooth curve of degree $d \geq 4$. Then, $|\mathrm{Aut}(C)| \leq 6d^2$ with "=" if and only if $C \simeq \{x_0^d + x_1^d + x_2^d = 0\}$ FERMAT. Unless:

① $d = 4 \rightsquigarrow C_{max} \simeq \{x_0^3 x_1 + x_1^3 x_2 + x_2^3 x_0 = 0\}$ KLEIN $\rightsquigarrow PSL_2(\mathbb{F}_7)$

② $d = 6 \rightsquigarrow C_{max} \simeq \{10x_0^3x_1^3 + 9x_0^5x_2 + 9x_1^5x_2 - 45x_0^2x_1^2x_2^2 - 135x_0x_1x_2^4 + 27x_2^6 = 0\}$
↳ "WIMAN sextic" $\rightsquigarrow A_6$

✓ Oguiso & Yu (2019): $X \subseteq \mathbb{P}^4$ quintic threefold $\rightsquigarrow$ Det. all admissible G (22 non-isomorphic of max. order)

✓ Adler (1978) + Wei & Yu (2020): $X \subseteq \mathbb{P}^4$ cubic threefold $\rightsquigarrow$ Det. all admissible G (6 non-isomorphic of max. order)

✓ Zheng (arXiv 2020): Det. ALL $G \simeq \mathbb{Z}/q\mathbb{Z}$ admissible in $\mathcal{U}_n(d)$

§3. A Liftability result

Dolgachev & Iskovskikh ⇝ Study $\mathrm{Aut}(S) \curvearrowright H^2(S,\mathbb{Z}) \cong \mathbb{Z}^7$

{ Oguiso & Wei & Yu ⇝ Study whether or not a group
{ $G \subseteq \mathrm{Aut}(X) \subseteq PGL_5(\mathbb{C})$ admits a **LIFTING** $\tilde{G} \subseteq GL_5(\mathbb{C})$

↙ our starting point! Representation Theory!

Let $X = \{x \in \mathbb{P}^{n+1} \text{ st } F(x) = 0\}$ as before, and let $G \subseteq \mathrm{Aut}(X) \subseteq PGL_{n+2}(\mathbb{C})$ be a subgroup. We say that a subgroup $\tilde{G} \subseteq GL_{n+2}(\mathbb{C})$ is a **LIFTING** of G if:

① $\pi|_{\tilde{G}} : \tilde{G} \subseteq GL_{n+2}(\mathbb{C}) \xrightarrow{\sim} G \subseteq PGL_{n+2}(\mathbb{C})$ isomorphism.

② For every $g \in \tilde{G}$ we have $g \cdot F = F$

$x_i \mapsto \xi x_i, \ \xi^d = 1$

Examples:

a) $X_F = \{x_0^d + \ldots + x_{n+1}^d = 0\} \subseteq \mathbb{P}^{n+1} \Rightarrow \mathrm{Aut}(X_F) \cong S_{n+2} \ltimes (\mathbb{Z}/d\mathbb{Z})^{n+1}$ liftable ✓

b) $X_K = \{x_0^{d-1}x_1 + x_1^{d-1}x_2 + \ldots + x_{n+1}^{d-1}x_0 = 0\} \subseteq \mathbb{P}^{n+1}$, and assume that $\gcd(d, n+2) > 1$. Given p prime st $p \mid d$ & $p \mid (n+2)$, we consider

$$\tilde{g} := \mathrm{diag}(\underbrace{1, \xi, \xi^2, \ldots, \xi^{p-1}, \ldots, 1, \xi, \xi^2, \ldots, \xi^{p-1}}_{(n+2)/p\text{ -times}}) \qquad \xi = e^{\frac{2\pi i}{p}}$$

$\Rightarrow \tilde{g}$ induces $g \in \mathrm{Aut}(X_K)$. However, $\tilde{g} \cdot K = \xi K$ is not liftable!

Thm A (GA. L. M. 2020): $X \subseteq \mathbb{P}^{n+1}$ smooth hypersurface of degree $d \geq 3$ st $(n,d) \notin \{(1,3),(2,4)\}$. Then:

$$\mathrm{Aut}(X) \text{ liftable} \iff \gcd(d, n+2) = 1.$$

⇑

Thm B (GA. L. M. 2020): Let $n \geq 1$ and $d \geq 3$ st $(n,d) \notin \{(1,3),(2,4)\}$ and let $q = p^r$ with $r \in \mathbb{N}^{\geq 1}$ and p prime. Then, q is the order of some liftable automorphism of some $X \in \mathcal{U}_n(d)$ iff

a) $p \mid (d-1)$ and $r \leq k(n+1)$, where $d-1 = p^k e$ & $p \nmid e$; or

b) $p \mid d$ and $\exists\, \ell \in \{1, \ldots, n+1\}$ st $(1-d)^\ell \equiv 1 \pmod q$; or

c) $p \nmid d(d-1)$ and $\exists\, \ell \in \{1, \ldots, n+2\}$ st $(1-d)^\ell \equiv 1 \pmod q$ ← [GA. L. 2011, 2013]

↳ An independent work of Z. Zheng (2020) generalizes this!

Cubic examples: Let $X \subseteq \mathbb{P}^{n+1}$ smooth cubic. Then, all the possible $\mathbb{Z}/p^r\mathbb{Z}$ which are admissible and liftable are:

- **Surfaces**: 2^{r_2} $(r_2 \leq 3)$, 3^{r_3} $(r_3 \leq 2)$ or 5.
- **Threefolds**: 2^{r_2} $(r_2 \leq 4)$, 3^{r_3} $(r_3 \leq 2)$, 5 or 11.
- **Fourfolds**: 2^{r_2} $(r_2 \leq 5)$, 3^{r_3} $(r_3 \leq 2)$, 5, 7 or 11.
- **Fivefolds**: 2^{r_2} $(r_2 \leq 6)$, 3^{r_3} $(r_3 \leq 2)$, 5, 7, 11 or 43.

As an application, we can get information about certain Sylow p-subgroups of $\mathrm{Aut}(X)$ (ie, of order p^r with r maximal)

Notation: Let p be a prime number st $p \nmid d(d-1)$ and let $r \in \mathbb{N}^{\geq 1}$. We define

$$\ell(p^r) := \min\{k \in \mathbb{N}^{\geq 1} \text{ st } (1-d)^k \equiv 1 \pmod{p^r}\}$$

Prop C (G.A.L.M. 2020): Let $X \subseteq \mathbb{P}^{n+1}$ smooth hypersurface of degree $d \geq 3$ st $\gcd(d, n+2) = 1$, and let p be a prime st $p \nmid d(d-1)$:

If $\ell(p^2) > n+2$ & $2\ell(p) > n+2 \implies p^2 \nmid |\mathrm{Aut}(X)|$.

In part, if $G \subseteq \mathrm{Aut}(X)$ p-Sylow then $G \simeq \{1\}$ or $\mathbb{Z}/p\mathbb{Z}$.

Cubic examples: Let $X \subseteq \mathbb{P}^{n+1}$ smooth cubic. Then:

$n=2$: $|\mathrm{Aut}(X)| = 2^{r_2} 3^{r_3} 5^{r_5}$ with $r_5 \leq 1$.

$n=3$: $|\mathrm{Aut}(X)| = 2^{r_2} 3^{r_3} 5^{r_5} 11^{r_{11}}$ with $r_5, r_{11} \leq 1$.

$n=4$*: $|\mathrm{Aut}(X)| = 2^{r_2} 3^{r_3} 5^{r_5} 7^{r_7} 11^{r_{11}}$ with $r_5, r_7, r_{11} \leq 1$

$n=5$: $|\mathrm{Aut}(X)| = 2^{r_2} 3^{r_3} 5^{r_5} 7^{r_7} 11^{r_{11}} 43^{r_{43}}$ with $r_5, r_7, r_{11}, r_{43} \leq 1$

Idea: Enough to analyze the cases

$G \simeq \mathbb{Z}/p^2\mathbb{Z}$ ← impossible by assumption, or $G \simeq (\mathbb{Z}/p\mathbb{Z})^2$ ← "not enough space"

§4. Sketch of Proof: Let $X = \{F = 0\} \subseteq \mathbb{P}^{n+1} = \mathbb{P}(V)$

smooth hypersurface, $F \in S^d(V^*)$ with $(n,d) \notin \{(1,3),(2,4)\}$.

Step 1 Consider $\varphi \in \mathrm{Aut}(X) \subseteq PGL(V)$ and assume $\mathrm{ord}(\varphi) = q$.

Let $\tilde{\varphi} \in GL(V)$ st $\pi(\tilde{\varphi}) = \varphi$ and $\tilde{\varphi}^q = \mathrm{Id}_V$.

[Key Rmk: Let p prime st $p \mid d$ and suppose $\mathrm{ord}(\varphi) = p$.
If φ is not liftable $\Rightarrow p \mid (n+2)$.

↳ Idea: $\tilde{\varphi} \cdot F = \xi F$, $\xi^p = 1$ and $\xi \neq 1$. $(\star)$

Let $V(i) \subseteq V$ be the eigenspace assoc. to ξ^i

$(\star)$ + X smooth $\Rightarrow \dim V(0) = \dim V(i)\ \forall i \Rightarrow n+2 = p \dim V(0)$ ✓

[Lemma: $\mathrm{ord}(\varphi) = q$ and $\tilde{\varphi} \cdot F = \xi^c F$ with $\xi^q = 1$ primitive
and $c \in \mathbb{Z}$. Then, φ liftable $\iff \gcd(d,q) \mid c$.

⚠ In part, if φ is not liftable: $\exists\, p$ prime factor of $\gcd(d,q)$
st $p \nmid c$. Write $q = pr$ and $\psi := \varphi^r$ of order p.

$\rightsquigarrow \varphi$ not liftable $\Rightarrow \psi$ not liftable $\Rightarrow \gcd(d, n+2) > 1$.

Step 2 Consider φ st $\mathrm{ord}(\varphi) = p^r$ and φ liftable

$\rightsquigarrow$ We prove Thm B by analyzing the eigenspaces of $\tilde{\varphi} = \mathrm{diag}(\xi^{\sigma_0}, \ldots, \xi^{\sigma_{n+1}})$

+ Providing explicit examples for the "if" part.

Step 3* Consider φ st $\mathrm{ord}(\varphi) = p^r$ and sup. $p \nmid d(d-1)$
or $p \nmid (n+2)$. Then, $\exists!$ lifting $\tilde{\varphi}$ to $SL(V)$.

↳ Application: X cubic 4-fold $\rightsquigarrow F(X) \underset{\text{defs}}{\simeq} K3^{[2]}$

$\varphi \in \mathrm{Aut}(X)$ induces $\hat{\varphi} \in \mathrm{Aut}(F(X))$. Then, for $\mathrm{ord}(\varphi) = p^r$
st $p \neq 2,3$ we have that $\hat{\varphi}$ symplectic $\Rightarrow \mathrm{ord}(\varphi) = 5, 7, 11$ (cf. Fu '16)

Step 4* Lifting of Sylow p-subgroups $G_p \leq \mathrm{Aut}(X)$ (to $SL(V)$)
in many cases !) → Here: We use $\gcd(d, n+2) = 1$ to simplify
and extend some arguments from group cohomology used by Oguiso-Yu
(e.g. Hochschild-Serre exact seq.) $\rightsquigarrow$ Get a lifting of $\mathrm{Aut}(X)$ to $GL(V)$ ■

§5. Some open questions:

?) **Schur multiplier:** Let G be a (finite) group.

$\rightsquigarrow M(G) := H^2(G, \mathbb{C}^*)$ Schur multiplier, allows to study projective representations $\rho: G \to PGL(V)$

Q: Given $X = \{x \in \mathbb{P}^{n+1} \text{ st } F(x) = 0\} \subseteq \mathbb{P}^{n+1}$ smooth hypersur, can we define a Schur mult relative to X with "nice" properties?

?) **Abelian subgroups of maximal order:** In a work in progress with Víctor GONZALEZ & Álvaro LIENDO, we are looking at abelian p-groups admissible in $\mathcal{U}_n(d)$

↳ 1° Complementary to the work of ZHENG (2020).

↳ 2° MUCH easier than general p-groups (eg. 99,2% of finite groups of order ≤ 2000 have order 2^{10})!

?) **Algebraic curves:** As far as I know, the possible automorphism groups of maximal order that arise as $Aut(C)$ for $C \subseteq \mathbb{P}^2$ plane curve of degree $d \geq 4$ are classified up to $d \leq 5$.

?) **Singular case:** How singular $X \subseteq \mathbb{P}^{n+1}$ can be in order that $Aut(X)$ is liftable? (cf. Hilbert-Mumford criterion in GIT).

Thank you for your attention!