NON-ARCHIMEDEAN APPROACH to MIRROR SYMMETRY and to DEGENERATIONS of VARIETIES

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Geometric explanation for MS

Ex: $V(f_4) \subseteq \mathbb{P}^3$ K3 surface
$V(f_5) \subseteq \mathbb{P}^4$ quintic 3-fold

variety with a nowhere vanishing holomorphic n-form,
equiv: with trivial canonical line bundle

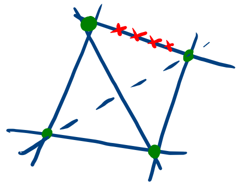
Consider a projective family of complex Calabi-Yau varieties of dim n
$X \longrightarrow \Delta^* \subseteq \mathbb{C}_t$ such that X is maximally degenerate

"meromorphic at 0": it extends to a projective family $\mathcal{X} \longrightarrow \Delta$
where $\mathcal{X}$ is smooth
and $\mathcal{X}_0$ is strict normal crossings

"as degenerate as poss": there is a non-empty intersection of $n+1$ comp's of $\mathcal{X}_0$

Ex: $\mathcal{X} = \{ t f_4 + x_0 x_1 x_2 x_3 = 0 \} \subseteq \mathbb{P}^3_x \times \mathbb{C}_t$

$\mathcal{X}_0$:



Geometric explanation for MS : SYZ conjecture

Consider a projective family of complex Calabi-Yau varieties of dim n

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Then a general fibre X_t admits a fibration $X_t \longrightarrow B$

- to a topological manifold B,
- whose fibres are special Lagrangian real tori of dim n
- away from a locus Δ of codim ≥ 2 in B

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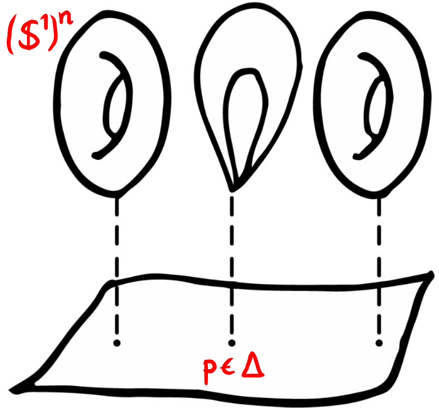
$\pi : X_t \longrightarrow B$



- construct $\check{X}_t$ from $\pi : X_t \to B$
- CYs form a mirror pair if they admit dual torus fibrations

$$\begin{tikzcd} X_t \arrow[dr, "\pi"'] & & \check{X}_t \arrow[dl, "\check{\pi}"] \\ & B & \end{tikzcd}$$

Base B of SYZ fibration

Given an SYZ fibration $\pi : X_t \longrightarrow B$

- B is a topological manifold of $\dim_{\mathbb{R}} B = n$

expectation: $\begin{cases} X_t \text{ strict CY of dim } n : & B \simeq S^n \\ X_t \text{ HK of dim } n \text{ (even)} : & B \simeq \mathbb{CP}^{\frac{n}{2}} \\ X_t \text{ abelian variety of dim } n : & B \simeq (S^1)^n \end{cases}$

- outside Δ, B admits an integral affine structure

defn: $B \setminus \Delta$ admits an open cover $(U_i)_i$ and charts $(\varphi_i : U_i \longrightarrow \mathbb{R}^n)_i$ such that the transition functions $\varphi_i \circ \varphi_j^{-1} \in GL_n(\mathbb{Z})$

- outside Δ, B admits a Monge-Ampère metric

Kontsevich-Soibelman insight: relate B to degenerate fiber $\mathcal{X}_0$ $\rightsquigarrow$ B as a dual complex embedded in a Berkovich space

metric limit (for B) · geometric limit (for $\mathcal{X}_0$)

Dual complexes

$X \longrightarrow \Delta^* \subseteq \mathbb{C}_t$

$\mathcal{X} \longrightarrow \Delta \subseteq \mathbb{C}$ snc degeneration of X

$\mathcal{X}_0$ snc

$\rightsquigarrow$

$D(\mathcal{X}_0)$ dual complex

$\parallel$

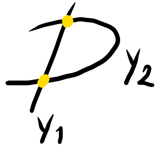
Given $Y = \bigcup Y_i$ snc variety, the dual complex is a cell complex consisting of

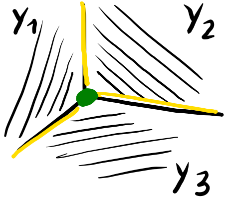
irr comp $Y_i \longleftrightarrow$ 0-cell v_i

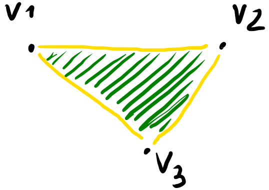
irr comp of $Y_i \cap Y_j \neq \emptyset \longleftrightarrow$ 1-cell $\langle v_i, v_j \rangle$

irr comp of $Y_{i_0} \cap \ldots \cap Y_{i_k} \neq \emptyset \longleftrightarrow$ k-cell $\langle v_{i_0}, \ldots, v_{i_k} \rangle$

Examples:

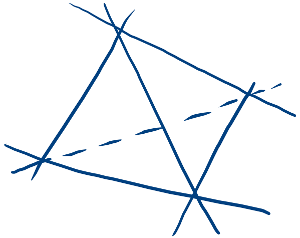
1) 

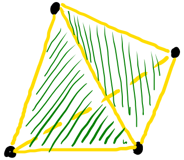
2) 



3) $\mathcal{X} = \{ t f_4 + x_0 x_1 x_2 x_3 = 0 \} \subseteq \mathbb{P}^3_x \times \mathbb{C}_t$

$\mathcal{X}_0 = (x_0 x_1 x_2 x_3 = 0) \subseteq \mathbb{P}^3_{\mathbb{C}}$





$\simeq S^2$

Dual complexes & Berkovich spaces

$v : K(X)^{\times} \longrightarrow \mathbb{R}$

$v(ab) = v(a) + v(b)$

$v(a+b) \geq \min\{v(a), v(b)\}$

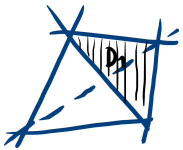
Ex: on $\mathbb{C}((t))$, $\mathrm{ord}_t\left(\sum_{n \geq n_0} a_n t^n\right) = n_0$

X smooth variety over $K = \mathbb{C}((t))$

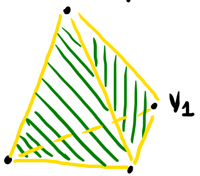
X^{an} Berkovich space of X $\supset$ $\{$valuations on $K(X)\}$

$\mathcal{X}$ snc degeneration of X over $\mathbb{C}[[t]]$

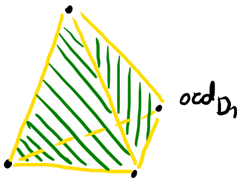
$\mathcal{X}_0 = \sum D_i$
snc divisor



$D(\mathcal{X}_0)$
dual complex of $\mathcal{X}_0$



$D(\mathcal{X}_0) \hookrightarrow X^{an}$
canonical embedding



$v_1 \longmapsto \mathrm{ord}_{D_1}$

locally: $D_1 = \{f_1 = 0\}$

$\mathrm{ord}_{D_1}(f) = \mathrm{ord}_{D_1}(f_1^a h) = a$

- image is $Sk(\mathcal{X})$ skeleton of $\mathcal{X}$
- retraction $\rho_{\mathcal{X}} : X^{an} \longrightarrow Sk(\mathcal{X})$

Berkovich analytification

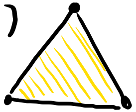
X smooth variety over $K = \mathbb{C}((t))$

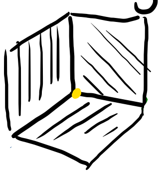
X^{an} Berkovich space of X

$$\begin{tikzcd} X^{an} \arrow[d, bend right, "\rho_{\mathcal{X}}"'] \\ Sk(\mathcal{X}) \arrow[u] \end{tikzcd}$$

$Sk(\mathcal{X}) \simeq D(\mathcal{X}_0)$ for $\mathcal{X}$ snc degeneration

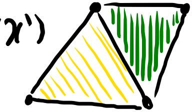
$Sk(\mathcal{X})$

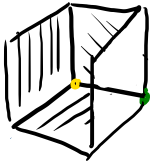




$\mathcal{X}_0$

$Sk(\mathcal{X}')$



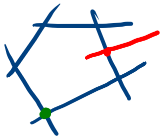


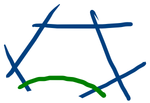
$\mathcal{X}_0'$

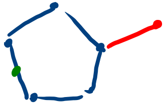
Prop: $X^{an} \simeq \varprojlim_{\mathcal{X} \text{ snc}} Sk(\mathcal{X})$

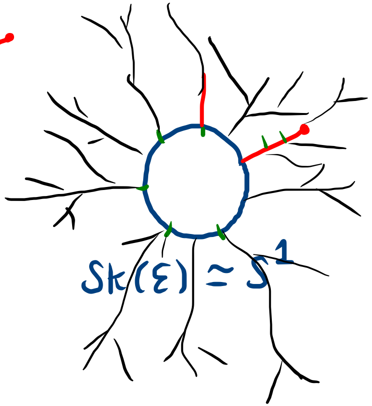
Ex: E elliptic curve over $k = \mathbb{C}((t))$ with minimal snc degeneration $\mathcal{E}$ s.t. $\mathcal{E}_0$ loop of rational curves

$\mathcal{E}_0$ $D(\mathcal{E}_0)$ E^{an}









$Sk(\mathcal{E}) \simeq S^1$

$\rho_{\mathcal{E}} : E^{an} \longrightarrow Sk(\mathcal{E})$

Essential skeleton

X smooth variety over $K = \mathbb{C}((t))$:

X^{an} Berkovich space of X $\supset$ $Sk(\mathcal{X})$ for any $\mathcal{X}$ snc degeneration $\supset$ $Sk^{ess}(X)$

- inspired by Kontsevich-Soibelman
- 1st approach : non-arch defn
- 2nd approach : birational construction

[Mustata-Nicaise, Nicaise-Xu]
- $Sk^{ess}(X)$ is a birational invariant of X
- $Sk^{ess}(X) = D(\mathcal{X}_{min,0})$ for any minimal dlt degeneration (generalization of min snc)
- $\rho_{\mathcal{X}_{min}} : X^{an} \longrightarrow Sk^{ess}(X)$ retraction (non-canonical)

[Brown-Mazzon] Let X be birational to $Hilb^n(S)$ or $K_n(A)$ (families of HK of dim 2n) where S K3 surface, A abelian surface, max degenerate. Then $Sk^{ess}(X)$ is homeomorphic to $\mathbb{CP}^n$

Rmk: [Kollár-Laza-Saccà-Voisin] Let $\mathcal{X}$ min dlt degeneration of 2n-dim HK, max degenerate, then $D(\mathcal{X}_{min,0})$ has $\mathbb{Q}$-homology of $\mathbb{CP}^n$

Non-archimedean SYZ fibration

X smooth CY variety over $K = \mathbb{C}((t))$

$X^{an} \supset Sk(\mathcal{X}) \supset Sk^{ess}(X)$

Ex: $\rho_E : E^{an} \longrightarrow Sk^{ess}(E) \simeq S^1$

SYZ conjecture | non-archimedean SYZ fibration

$$\begin{array}{ccccccc} X_t & & X^{an} & & (\mathbb{G}_m^n)^{an} & \ni & v \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho_{\mathcal{X}_{min}}} & \text{locally isomorphic} & \downarrow{\scriptstyle trop} & & \downarrow \\ B & \simeq D(\mathcal{X}_{min,0}) = & Sk^{ess}(X) & & \mathbb{R}^n & & (v(z_i)) \end{array}$$

[Nicaise-Xu-Yu] For any $\mathcal{X}_{min}$ good minimal dlt degeneration of X the retraction $\rho_{\mathcal{X}_{min}} : X^{an} \longrightarrow Sk^{ess}(X)$ is an affinoid torus fibration away from a locus of codim ≥ 2

[Mazzon-Pille-Schneider] in preparation: For degenerations of quartic K3 surfaces and quintic 3-folds (strict CY) by non-archimedean SYZ fibration, $Sk^{ess}(X) \simeq S^n$ can be endowed with an integral affine structure equal to the one classically constructed on B in mirror symmetry