On the unirationality of Quadric bundles and Hypersurfaces

X projective variety over a field k is:

- rational if there is a birational map

$$\mathbb{P}^n \dashrightarrow X$$

- unirational if there is a dominant map

$$\mathbb{P}^n \dashrightarrow X$$

- stably rational if $X \times \mathbb{P}^r$ is rational for some $r \geq 0$
- rationally connected if any two points of X can be joined by a rational curve

X rational $\underset{\not\Leftarrow}{\Rightarrow}$ X is stably rational $\underset{\not\Leftarrow}{\Rightarrow}$ X unirational $\underset{\overset{?}{\Leftarrow}}{\Rightarrow}$ X is RC

[BC] Châtelet surfaces
To construct a complex conic bundle Y s.t. Y is not rational but $Y \times \mathbb{P}^3$ is rational

Several hypersurface in $\mathbb{P}^N$
[K], [T], [S]

MMP $\Rightarrow$ A variety covered by rational curves is birational equivalent to a mildly singular variety Z with a contraction $\pi : Z \longrightarrow W$ of relative Picard number 1 s.t. $-K_Z$ is π-ample. When $\dim(Z) = 3$:

(1) W is a point and Z is Fano;

(2) W is a curve and Z is a del Pezzo fibration;

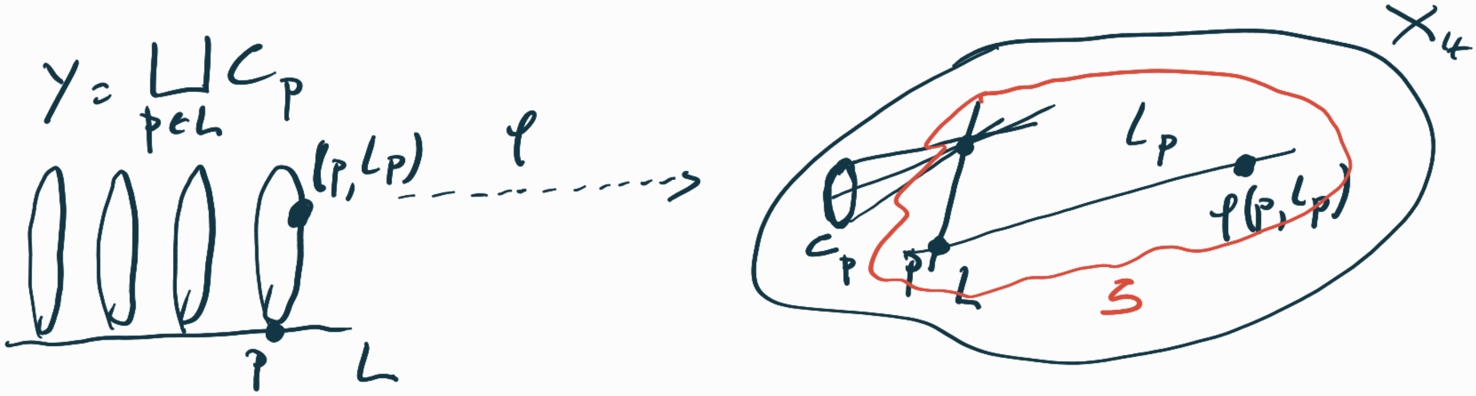
(3) W is a surface and $Z \to W$ is a conic bundle.

(1) Most of these are unirational. It is not known whether the general sextic double solid ($Z \to \mathbb{P}^3$ ramified over a sextic) is unirational, and unirationality is not known for the general quartic 3-fold.

(3) [KM] A conic bundle $X \to \mathbb{P}^2$ with discriminant of degree at most 8 is unirational

Remark) There are examples of smooth unirational quartic 3-folds (Segre)

$X_4 \subseteq \mathbb{P}^4$ is Fano $\Rightarrow$ Rationally Connected, Take a rational curve $L \subseteq X_4$



Y is a surface conic bundle (over $\mathbb{C}$) Y is rational

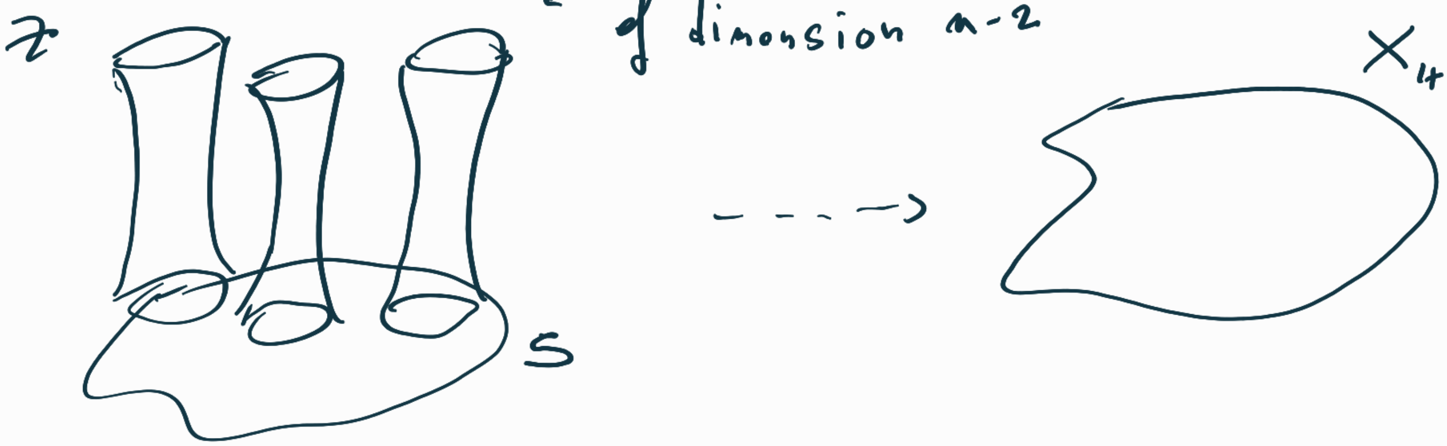
$S = \varphi(Y) \subseteq X_4$ is a rational surface.

Redo this construction replacing L with S

If the 3-conic bundle Z is unirational then X_4 is unirational.

Problem) The discriminant of $Z \to S$ is very big.

$X_4^n \subset \mathbb{P}^{n+1}$



Lang's theorem tells you that $Z \to S$ has a section

if $n \geq 5 \Rightarrow Z$ is rational $\Rightarrow$ A general quartic hypersurface of dimension at least 5 is rational (over $\mathbb{C}$).

Theorem) $X_4^n \subseteq \mathbb{P}^{n+1}$ quartic, $\Lambda \subseteq \mathbb{P}^{n+1}$ an h-plane.
Assume that either

(i) $n \geq 3$; $h \geq 2$; $\dim(\Lambda \cap \mathrm{Sing}(X_4^n)) \leq h-2$, X_4 contains Λ;

(ii) $n \geq 4$; $h \geq 3$; X_4^n has double points along Λ a point $p \in X_4^n \setminus \Lambda$;

and X_4^n is otherwise general;

X_4^n is unirational.

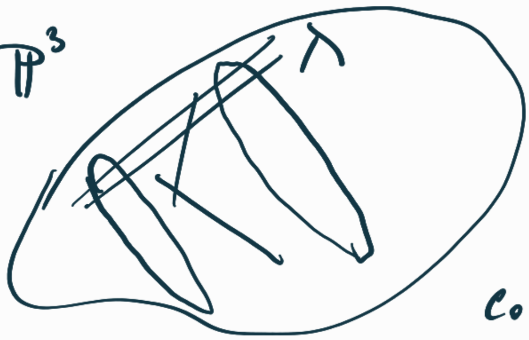
Questions)

(i) $n=3$, X_4 contains a line;

(ii) $n=3$, X_4 has double points along Λ and $\dim(\Lambda)=1,2$.

(iii) $n=2$, X_4 has double points along a line and a point $p \in X_4 \setminus \Lambda$.

(iii) $\mathbb{P}^3$



projection from Λ $\dashrightarrow$ $\mathbb{P}^1$

conic bundle with 8 singular fibers.

Theorem (KM) Surface conic bundles with at most 7 singular fibers are unirational.

Fix $a_0, \dots, a_{h+1} \in \mathbb{Z}_{\geq 0}$ $\quad a_0 \geq \dots \geq a_{h+1}$

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-h}}(a_0) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^{n-h}}(a_{h+1})) \longrightarrow \mathbb{P}^{n-h}$$

This is a toric variety with Cox ring

$$Cox = K[x_0, \dots, x_{n-h}, y_0, \dots, y_{h+1}]$$

irrelevant ideal $(x_0, \dots, x_{n-h}) \cap (y_0, \dots, y_{h+1})$

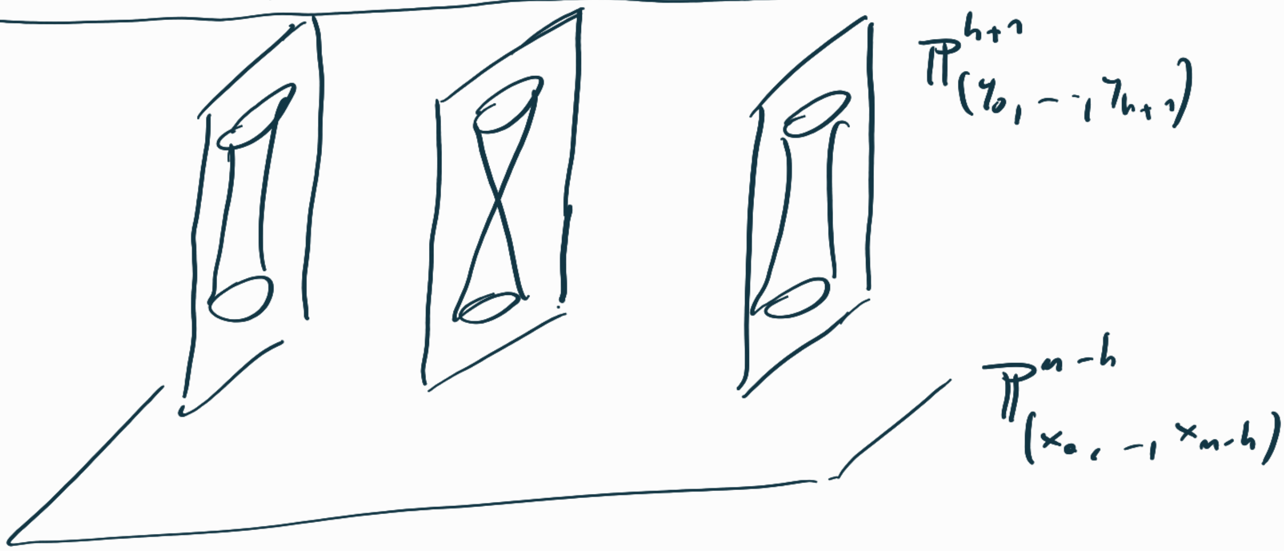
and grading

$$\begin{array}{ccccccc} x_0 & & x_{n-h} & y_0 & & y_{h+1} \\ \end{array}$$
$$\begin{pmatrix} 1 & \dots & 1 & -a_0 & \dots & -a_{h+1} \\ 0 & & 0 & 1 & & 1 \end{pmatrix}$$

$$Q^h = \Big\{ \sum_{0 \leq i \leq j \leq h+1} \sigma_{i,j}(x_0, \dots, x_{n-h})\, y_i y_j = 0 \Big\}$$

$d_{i,j} = \deg \sigma_{i,j}$

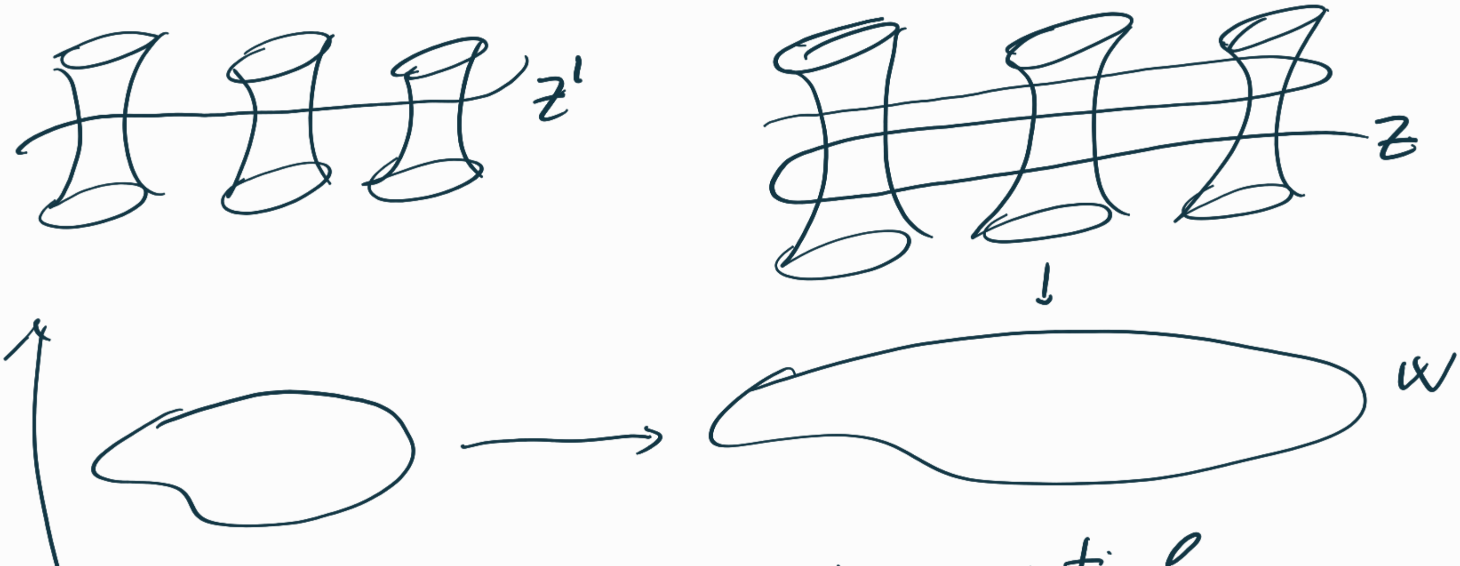
$$\boxed{d_{0,0} - 2a_0 = d_{0,1} - a_0 - a_1 = \dots = d_{h+1,h+1} - 2a_{h+1}}$$



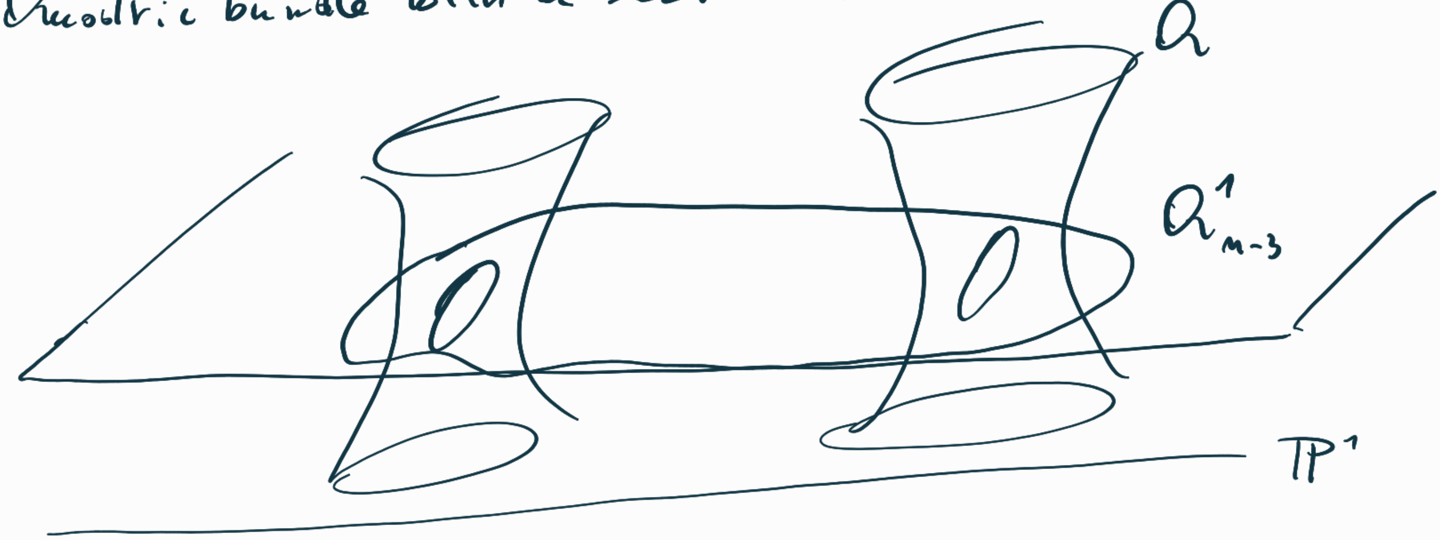
The locus $\mathbb{P}^{n-h}$ over which the quadrics do not have maximal rank is the discriminant of the quadric bundle.

Prop (Enriques's Criterion) $\pi: Q^h \to W$ fibration is quadrics over a unirational variety W. Then Q^h is unirational $\iff$ there is a unirational variety $Z \subseteq Q^h$ s.t. $\pi_{|Z}: Z \to W$ is dominant.

$$\begin{array}{ccc} \mathbb{P}^n & \dashrightarrow & Q^h \supseteq Z \\ \rotatebox{90}{$\supseteq$}\, H & & \downarrow \\ \text{general } (n-h)\text{-plane} & & W^{n-h} \end{array}$$



Quadric bundle with a section $\Rightarrow$ rational.



$$Q^1_{n-3} = Q^h \cap (Y_0 = Y_1 = \dots Y_{n-3} = 0)$$

If $\underset{\text{odd}}{\delta_{Q^h}} \leq 4m-1 \Rightarrow \underset{\text{odd}}{\delta_{Q^1}} \leq 7$

with a single exception which is $d_{m-2,m-2} = d_{m-1,m-1} = d_{m,m} = 3$

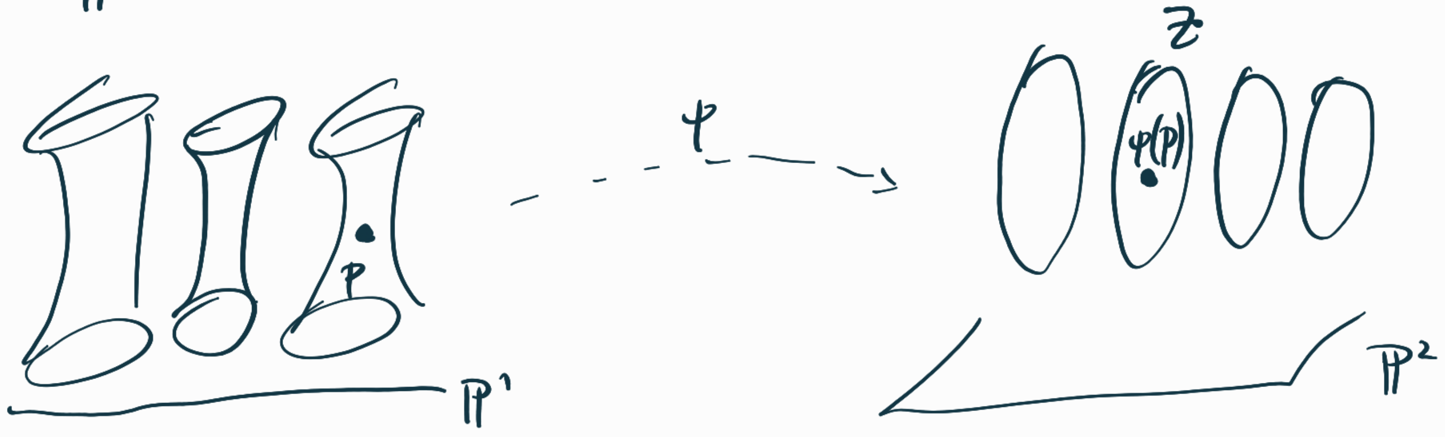
If $\delta_{Q^h} \leq 4m-1$, odd then $\delta_{Q^1} \leq 7$ odd. So we produced inside Q^h a transverse unirational variety Q^1 (unirational by [KM]).

Divisor of bidegree $(3,2)$ in $\mathbb{P}^1 \times \mathbb{P}^m$

Ottem

$$\begin{tikzcd} Q^{m-1} \arrow[r, dashed, "\varphi"] \arrow[d] & Q^{m-1}_+ \arrow[d, "\tilde{\pi}"] \\ \mathbb{P}^1 & \mathbb{P}^2 \end{tikzcd}$$

The fibers of $\tilde{\pi}$ are complete intersections of two quadrics.



Z is a complete intersection of two quadrics with a points and it is unirational.

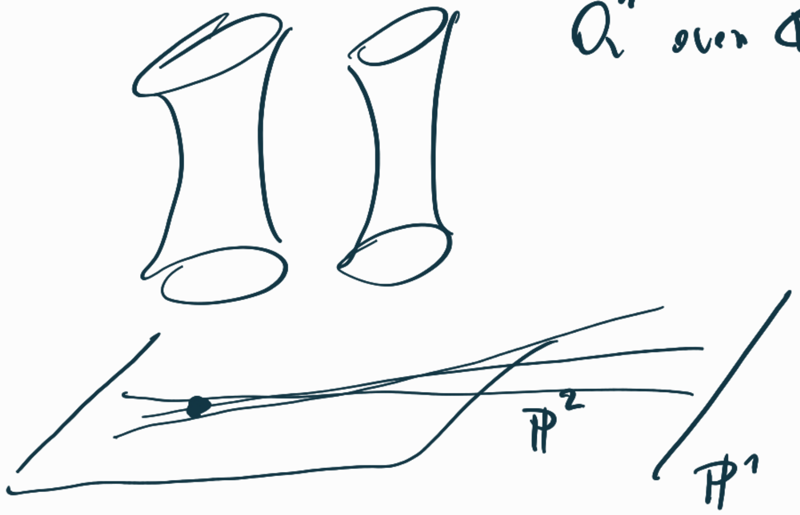
Th) $Q^{n-1} \longrightarrow \mathbb{P}^1$ a quadric bundle such $(-K_{Q^{n-1}})^n > 0 \Leftrightarrow (\delta_{Q^{n-1}} \leq 4n-1)$ and $\delta_{Q^{n-1}}$ is odd.

If either

(i) $n \leq 5$, Q^{n-1} has a point;

(ii) k is a number field.

Then Q^{n-1} is unirational.



Q^h over $\mathbb{C}$

The generic fiber Q^h_η is a quadric bundle over $\mathbb{P}^1_{\mathbb{C}(t)}$

If Q^h_η is unirational over $\mathbb{C}(t)$ then Q^h is unirational over $\mathbb{C}$.

Cor) $Q^2 \longrightarrow \mathbb{P}^2$ over $\mathbb{C}$, fibration in two-dimensional quadrics. If $\delta_{Q^2} \leq 12$ then Q^2 is unirational (for any smooth quadric bundle Q^2).