Convexity in tropical spaces and compactifications of cluster varieties

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Joint work with Man-Wai Cheung and Alfredo Nájera Chávez
arXiv:1912.13052 [math.AG]

Goal and Background

The Goal

Generalize the polytope construction of projective toric varieties to the non-toric world

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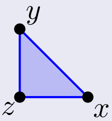
- Integral points of polytope $P \rightsquigarrow$ Sections of line bundle $\mathcal{L}$
- Integral points of dilations of $P \rightsquigarrow$ Sections of powers of $\mathcal{L}$

Convexity is key!

Example



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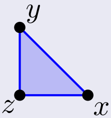

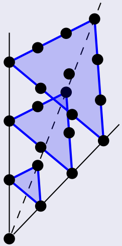

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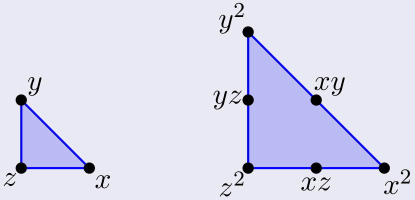

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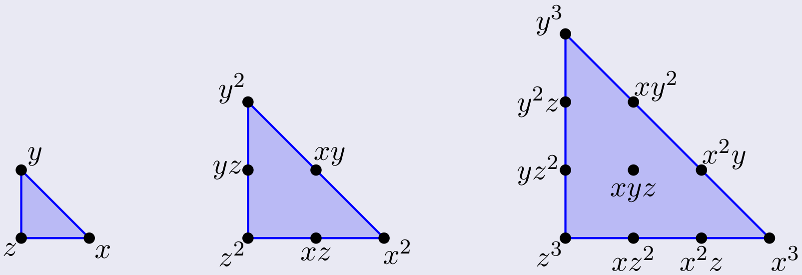

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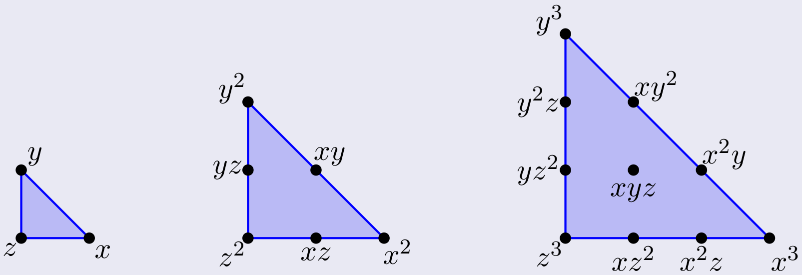
Example $(\mathbb{P}^2, \mathcal{O}(1))$



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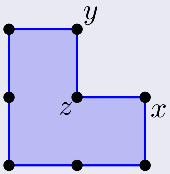

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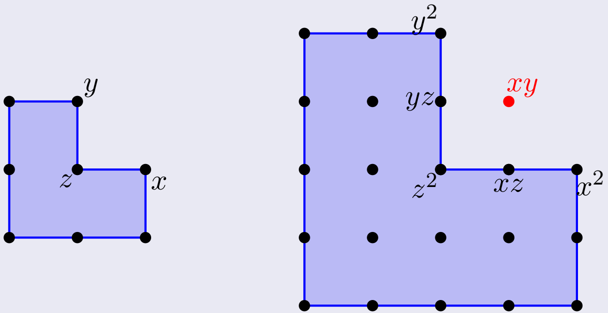

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Let $T \cong (\mathbb{C}^*)^n$ and $M = \mathrm{char}(T)$.

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Definition (*Log Calabi-Yau variety*)

A smooth complex variety U with a unique volume form Ω having at worst a simple pole along any divisor in *any* compactification of U

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Carefully glued tori

$$U = \bigcup_i T_i / \sim$$

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Cluster variety

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- Simpler than general Log CYs. ("Correct" setting should be *affine log CYs with maximal boundary*)
- Machinery isn't fully developed for general Log CYs yet.
- Interesting in many areas of math: Representation theory, Integrable systems, Hyperbolic geometry, Quantum groups, Scattering amplitudes. . .

Scattering Diagrams and Broken Lines

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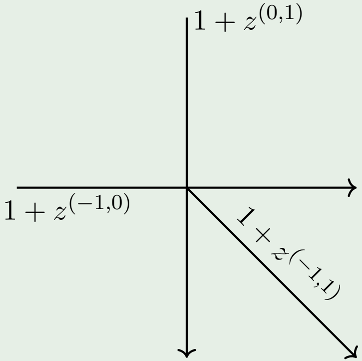
Wall: Codim 1 rational convex cone, decorated with a function (*scattering function*)
Used to construct a canonical algebra– determined by logarithmic geometry of U– which is conjectured to be $\mathcal{O}(U^{\vee})$ when U is an "affine log CY with maximal boundary".

Scattering Diagrams and Broken Lines

Example

$Y =$ del Pezzo surface of degree 5
$D =$ anticanonical pentagon in Y
$U = Y \setminus D$



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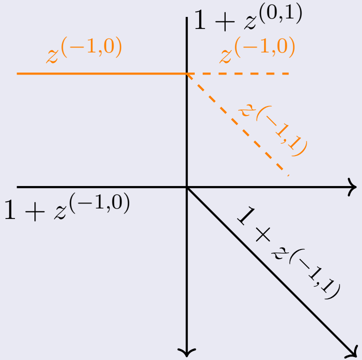

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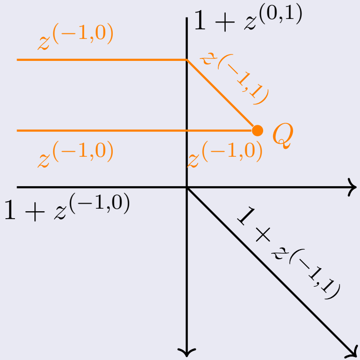
$$\vartheta_{(-1,0)} = z^{(-1,1)} + z^{(-1,0)}$$



ϑ-function multiplication

Structure constants α_{pq}^r

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Theorem (Gross-Hacking-Keel-Kontsevich)

$$\alpha_{pq}^r = \sum_{\substack{(\gamma_1,\gamma_2) \\ I(\gamma_1)=p,\ I(\gamma_2)=q \\ \gamma_1(0)=\gamma_2(0)=r \\ F(\gamma_1)+F(\gamma_2)=r}} c(\gamma_1)\ c(\gamma_2)$$

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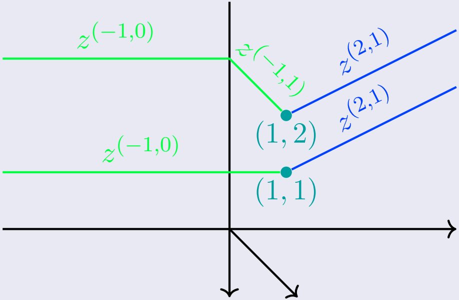

$$\vartheta_{(-1,0)} \cdot \vartheta_{(2,1)} = \vartheta_{(1,1)} + \vartheta_{(1,2)}$$

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So far have:

- $U^{\mathrm{trop}}(\mathbb{Z})$ "is" ϑ-basis for $\mathcal{O}(U^\vee)$

Convexity in $U^{\mathrm{trop}}(\mathbb{R})$

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Let's make this more precise.

Definition (Positive subset)

A closed subset $S \subset U^{\mathrm{trop}}(\mathbb{R})$ is **positive** if for every $a, b \in \mathbb{Z}_{\geq 0}$, $p \in aS(\mathbb{Z})$, $q \in bS(\mathbb{Z})$, and $r \in U^{\mathrm{trop}}(\mathbb{Z})$ with $\alpha^r_{p,q} \neq 0$ we have: $r \in (a+b)\, S$.

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When is S positive?

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Definition (Broken line convex [Cheung, M., Nájera Chávez])

A closed subset $S \subset U^{\mathrm{trop}}(\mathbb{R})$ is **broken line convex** if for every $x, y \in S(\mathbb{Q})$, every broken line segment connecting x and y is entirely contained in S.

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S is positive if and only if S is broken line convex.

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The generalization:

- $U^{\text{trop}}(\mathbb{Z})$ "is" ϑ-basis for $\mathcal{O}(U^{\vee})$
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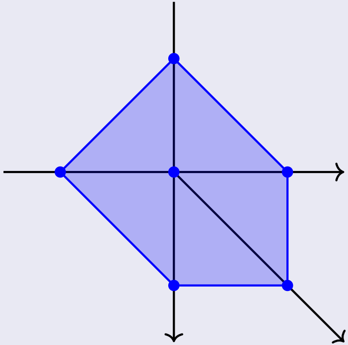
- The $U^{\text{trop}}(\mathbb{Z})$-points of S and its dilations give a basis for the section ring of $\mathcal{L}$

Examples

Example (Anticanonical "polytope" of degree 5 del Pezzo surface)



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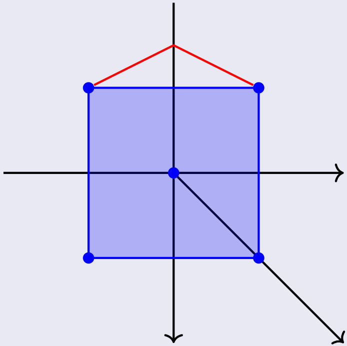

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- Choice of torus chart gives identification of $\Delta_{\mathrm{BL}}(D)$ with a usual Newton-Okounkov body.

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- Analogous result holds for complete flag variety

Basic Definitions

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Definition (Reflexive polytope)

A lattice polytope $P \subset M_{\mathbb{R}}$ is **reflexive** if its dual

$$P^{\circ} := \{n \in N_{\mathbb{R}} : \langle m, n \rangle \geq -1 \text{ for all } m \in P\}$$

is also a lattice polytope.

Review of Toric Case

Polytopes and toric Fanos

- If X is a d-dimensional Gorenstein Fano toric variety, then P_{-K_X} is a d-dimensional reflexive polytope.

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- Sections of $\mathcal{O}_X(D)$ and $\mathcal{O}_Y(D')$ are mirror CYs.

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Landau-Ginzburg Mirror and Anticanonical "Polytope"

Let (X, D) be a Fano minimal model of U, with $D = \sum_i D_{\nu_i}$.

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- We have the evaluation pairing:

$$\begin{aligned} \langle \cdot \, , \, \cdot \rangle : U^{\mathrm{trop}}(\mathbb{Z}) \times (U^\vee)^{\mathrm{trop}}(\mathbb{Z}) &\to \mathbb{Z} \\ (\ \nu \ , \ p\ ) \qquad &\mapsto \nu(\vartheta_p) \end{aligned}$$

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$\mathrm{Newt}_\vartheta(W)$ defines a minimal model (Y, D') of $U^\vee$.
Guess: Generic sections of $\mathcal{O}_X(D)$ and $\mathcal{O}_Y(D')$ are mirror (mildly singular) Calabi-Yau varieties.

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