Action of the Cremona group on a CAT(0) cube complex

Algebraic Geometry Seminar of the University of Nottingham

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joint work with Christian Urech

Cremona group of rank n

Let k be any field.

The **Cremona group** of rank n, denoted by $\mathrm{Bir}(\mathbb{P}^n_k)$, is the group of birational transformations of $\mathbb{P}^n$.

- $f \in \mathrm{Bir}(\mathbb{P}^n_k) \rightsquigarrow \quad f : \begin{array}{ccc} \mathbb{P}^n & \dashrightarrow & \mathbb{P}^n \\ [x_0 : \ldots : x_n] & \mapsto & [f_0 : \ldots : f_n], \end{array}$

  where $f_i \in k[x_0, \ldots, x_n]$ homogeneous of same degree and without common factor + f^{-1} of the same form.

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- $\deg(f) = \deg(f_i)$.
- $\cap\{f_i = 0\}$ set where f is not well defined.

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- $\deg(f) = \deg(f_i)$.
- $\cap\{f_i = 0\}$ set where f is not well defined.
- **Remark:** $\mathrm{Bir}(\mathbb{P}^n_k) \simeq \mathrm{Aut}_k\, k(x_1, \ldots, x_n)$.

Examples

- $\mathrm{Aut}(\mathbb{P}^n) = \{f \in \mathrm{Bir}(\mathbb{P}^n) \mid \deg(f) = 1\} \simeq \mathrm{PGL}_{n+1}(k)$.

- $\mathrm{GL}_n(\mathbb{Z})$: subgroup of monomial transformations.
  $(m_{ij})_{1\leqslant i,j\leqslant n} \leftrightsquigarrow (x_1, \ldots, x_n) \mapsto (\prod_{j=1}^n x_j^{m_{1j}}, \ldots, \prod_{j=1}^n x_j^{m_{nj}})$

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  - For instance the standard quadratic involution of Cremona: $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \leftrightsquigarrow (x, y) \overset{\sigma}{\mapsto} (1/x, 1/y)$.

    $[x:y:z] \dashrightarrow [yz:xz:xy]$

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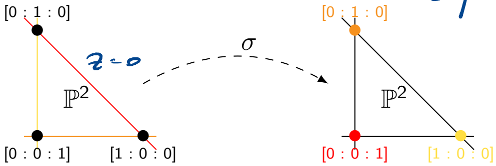
[yz : xz : xy]



Introduction

- L. Cremona introduced $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ in 1863-1865.
- Examples of field related to this group:
  - Algebraic geometry,
  - Complex, real, ... geometry
  - Dynamic,
  - Topology,
  - Geometric group theory...

Context: $\mathrm{Bir}(\mathbb{P}^2_k)$, a lot of results

- Generators:

Theorem ($\simeq$ 1900; Noether-Castelnuovo)

Let $k = \bar{k}$. $\mathrm{Bir}(\mathbb{P}^2) = < \mathrm{Aut}(\mathbb{P}^2), \sigma >$.

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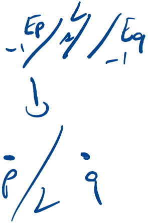
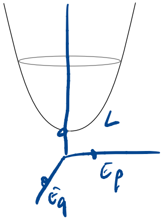
Let $k = \bar{k}$. $\mathrm{Bir}(\mathbb{P}^2) = < \mathrm{Aut}(\mathbb{P}^2), \sigma >$.

- Key tool to study $\mathrm{Bir}(\mathbb{P}^2_k)$:

$\mathrm{Bir}(\mathbb{P}^2) \curvearrowright \mathbb{H}^\infty$ $\overset{\text{analoguous}}{\leftrightsquigarrow}$ $\mathbb{H}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 - y^2 - z^2 = 1 \text{ and } x > 0\}$

$\mathbb{H}^\infty = \{c \in \mathcal{Z} \mid c \cdot c = 1 \text{ and } c \cdot \ell > 0\}$ $\subset \mathbb{R}^\infty$
where $\mathcal{Z}$ is $\varinjlim_{S \to \mathbb{P}^2} \mathrm{NS}(S) \otimes \mathbb{R}$





$\mathbb{P}^2$ $\mathrm{NS}(\mathbb{P}^2) = \langle L \rangle$

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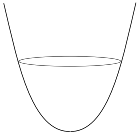


Theorem ($k = \bar{k}$ Cantat-Lamy ('13); k finite+... Shepherd-Barron ('21) ; k, L. ('16))

The Cremona group $\mathrm{Bir}(\mathbb{P}^2)$ is not simple.

Context: in higher rank, more complicated !

- Generators: No (non-trivial) system of generators known.

Theorem ('99 ; Pan)

For $n \geqslant 3$, $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$ can not be generated by elements of bounded degree.

- No (interesting) action known on geometric space.

Theorem ('20 ; Blanc-Lamy-Zimmermann)

For $n \geqslant 3$, the Cremona group $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$ is not simple.

Result

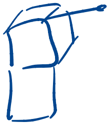
Theorem ('20, L.-Urech)

For all $n \geqslant 1$ and for any field k, we constructed a CAT(0) cube complex on which $\mathrm{Bir}(\mathbb{P}^n_k)$ acts non trivially.

$\rightsquigarrow$ for any group of birational transformations and for its subgroups of pseudo-automorphisms.

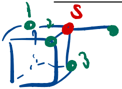
Definitions

- A **cube complex** is a union of unit euclidean cubes glued together along faces.
  Its **dimension** is the maximal dimension of its cubes, if it is finite.
- Remark: Two distances:
  * the one induced by the Euclidean metric.
  * the one in the 1-skeleton.

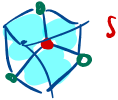


Definitions

- A **cube complex** is a union of unit euclidean cubes glued together along faces.
- Remark: Two distances:
  - ∗ the one induced by the Euclidean metric.
  - ∗ the one in the 1-skeleton.
- A cube complex C is **CAT(0)** if:
  - ∗ it is simply connected,
  - ∗ the links of its vertices are flag: $\forall s$ vertices of C, $\forall \{s_1, \dots, s_n\}$ vertices of C adjacent to s and pairwise in a square, $\{s, s_1, \dots, s_n\}$ belong to a cube.
- Example:



Non-example



Tools

- Tool 1: If G acts on an oriented CAT(0) cube complex with a bounded orbit, it fixes a vertex.
- Tool 2:

Theorem (Haglund)

Let G act on an oriented CAT(0) cube complex. For any $g \in G$

- either g fixes a vertex (elliptic).
- or g preserves a combinatorial geodesic and acts by translation on it.

Isomorphisms in codimension ℓ.

- Let X, Y variety. A birational transformation $f : X \dashrightarrow Y$ is an **isomorphism in codimension ℓ** if $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f^{-1})$ have codimension $> \ell$.

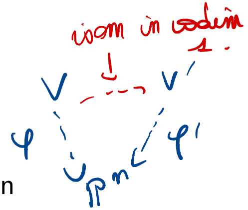
  $\rightsquigarrow$ $\mathrm{Psaut}^{\ell}(X)$ the group of automorphisms in codimension ℓ.

- Remark:
  - $\mathrm{Psaut}^0(X) = \mathrm{Bir}(X)$.
  - $\mathrm{Psaut}^n(X) = \mathrm{Aut}(X)$ where $n = \dim(X)$.
  - $\mathrm{Psaut}^1(X)$ group of pseudo-automorphisms.

- Example: If $D \subset X$ is a closed subvariety of codimension $\ell + 1$ then $\iota : X \setminus D \hookrightarrow X$ is an isomorphism in codimension ℓ.

Construction of the complex

Construction of $\mathcal{C}^0(\mathbb{P}^n)$



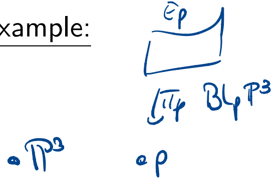
- **Vertices:** $[(V,\varphi)]$
  - $*$ V normal rational variety of dim n,
  - $*$ $\varphi : V \dashrightarrow \mathbb{P}^n$ birational transformation,
  - $*$ $(V,\varphi) \sim (V',\varphi')$ iff $\varphi'^{-1}\varphi$ is an isomorphism in codimension 1.

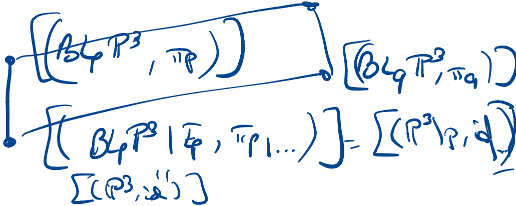
- Example:

$(\mathbb{P}^3, id) \sim (\mathbb{P}^3 \setminus \{pt\}, i)$

- **Edges:** $[(V,\varphi)]\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet[(V\setminus D, \varphi\,|_{V\setminus D})]$ where D is an irreducible subvariety of codim 1.

- Example:

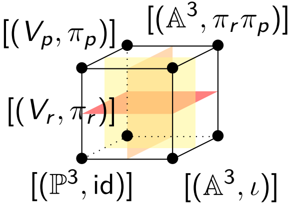




Construction of $\mathcal{C}^0(\mathbb{P}^n)$

- ***n*-cubes:** $[(V_1, \varphi_1)], \ldots, [(V_{2^n}, \varphi_{2^n})]$, if there exists $1 \leqslant r \leqslant 2^n$ such that for all $1 \leqslant i \leqslant 2^n$:
  - $D_1, \ldots, D_n \subset V_r$ irreducible, distinct of codim 1,
  - $V_i = V_r \setminus \{D_{i_1} \cup \cdots \cup D_{i_j}\}$
  - φ_i is the restriction of φ_r to V_i.

- Example:



Construction of the complex

Properties

- The complex is not locally finite.
- The complex is not of finite dimension.
- But it is oriented !
- Bir(X) acts on $\mathcal{C}^0(X)$:

$$f \bullet [(V, \varphi)] = [(V, f\varphi)].$$

Theorem ('20 ; L.-Urech)

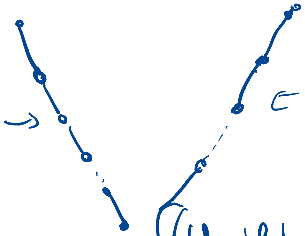
$\mathcal{C}^0(X)$ is a CAT(0) cube complex.

$\rightsquigarrow$ Consequence of the fact that any finite subset of vertices lies in a cube.

Geodesics

$[(V_1, \varphi_1)]$ $\quad$ $[(V_2, \varphi_2)]$

$\# \mathrm{Exc}^1(\varphi_2^{-1}\varphi_1)$ $\quad$ $\# \mathrm{Exc}^1(\varphi_1^{-1}\varphi_2)$



$[(U_1, \varphi_1|_{U_1})] = [(U_2, \varphi_2|_{U_2})]$

$\varphi_2^{-1}\varphi_1 : V_1 \dashrightarrow V_2$

$\mathrm{Exc}^1(\varphi_2^{-1}\varphi_1)$ ← irreducible components of codim 1 of the except locus

$\underbrace{V_1 \setminus \mathrm{Exc}^1(\varphi_2^{-1}\varphi_1)}_{U_1} \dashrightarrow \underbrace{V_2 \setminus \mathrm{Exc}^1(\varphi_1^{-1}\varphi_2)}_{U_2}$

isom in codim 1.

Results of (pseudo)-regularisation

- A subgroup $G \subset \mathrm{Bir}(X)$ is **(pseudo-)regularisable** if there exist a variety Y and $\varphi : Y \dashrightarrow \mathbb{P}^n$ such that

$$\varphi^{-1} G \varphi < (\mathrm{Ps})\,\mathrm{Aut}(Y).$$

Proposition ('20 ; L.-Urech)

Let $G < \mathrm{Bir}(X)$. G is pseudo-regularisable iff there exists $A \geqslant 0$ such that $\{Exc^1(g) \mid g \in G\} \leqslant A$.

Results of (pseudo)-regularisation

- A group has the **FW property** if every action on a CAT(0) cube complex has a fixed point.

$GL_2(\mathbb{Z}) \subset \mathrm{Bir}\,\mathbb{P}^3$

- Example: $\mathrm{SL}(2,\mathbb{Z})$, $\mathrm{SL}(2,\mathbb{Z}[\sqrt{3}])$...

Theorem ('20, L.-Urech ; Cornulier)

Let $G \subset \mathrm{Bir}(\mathbb{P}^n)$ which has the FW property, then G is regularisable. If moreover $n = 2$, G is projectively regularisable.

$\rightsquigarrow$ Answer a question of Cantat-Cornulier ('19 pseudo-regularisable).

$\rightsquigarrow$ Reprove immediately their result.

Construction of $\mathcal{C}^\ell(X)$ st $P_{Jou}^{\ell}(X) \simeq \mathcal{C}^\ell(X)$

- **Vertices:** $[(V, \varphi)]$
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Proof

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- $\mathrm{Psaut}^{\ell}(X)$ acts on $\mathcal{C}^{\ell}(X)$:

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~~$\rightsquigarrow$ Consequence of the fact that any finite subset of vertices lies in a cube.~~

Proof

$G < \mathrm{Bir}\,\mathbb{P}^2$ FW

$G \curvearrowright \mathcal{C}^0(\mathbb{P}^n)$ G fixes a vertex

$\Rightarrow \exists\, \varphi_1, V_1$ st $\varphi_1 : V_1 \dashrightarrow \mathbb{P}^n$

st $\varphi_1^{-1} G \varphi_1 < \mathrm{Psaut}^1(V_1)$

Now $\varphi_1^{-1} G \varphi_1 \curvearrowright \mathcal{C}^1(V_1)$ FW

--- $\exists\, \varphi_2, V_2$ st

$\varphi_2^{-1}\varphi_1^{-1} G \varphi_1 \varphi_2 < \mathrm{Psaut}^2(V_2)$

$\vdots$

$< \mathrm{Psaut}^n(V_n)$ (Aut)

Thanks!