On Miyaoka and Kawamata-Miyaoka type inequalities

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$/\mathbb{C}$ proj J.w.w. J. Liu

J.w.w. M. Iwai. & C. Jiang

§1. Main result.

Def. If a normal proj var ($-K_X$ is $\mathbb{Q}$-Cartier) X satisfies:

① $\mathbb{Q}$-factorial;

② $\rho(X)=1$;

③ $-K_X$ is ample.

then X is called $\underline{\mathbb{Q}\text{-Fano}}$. Only ③. then X $\underline{\text{Fano}}$.

only ③$'$. ($-K_X$ is nef and big). then X $\underline{\text{weak Fano}}$.

$\mathbb{Q}$-Fano $\subset$ Fano $\subset$ weak Fano.

Rmk. · singularities. (terminal. canonical. klt. lc ⋯)

· In the singular case. Chern Class is welled defined if it can be defined on the smooth locus and extend to the whole.

For example. smooth in codim k. then

$$c_i(X) = c_i(T_X) \text{ is well defined for } i \leq k.$$
$$\in A^i(X)$$

In particular $\underset{\text{terminal}}{\underline{\text{sm in codim } 2}}$. $c_1(X)$ $c_2(X)$. well, def.

Thm 1. (Miyaoka type inequality. Iwai.-L.-Jiang 23). Let X be a terminal. weak Fano var of dim n. then

$$c_2(X)c_1(X)^{n-2} > 0.$$

Thm 2 (KM type ineq for weak Fano. Iwai-L.-Jiang 23). Let X be a terminal. weak Fano. var of dim n. then $\exists$ $\underset{>0}{b_n}$ depending only on n. s.t.

$$c_1(X)^n \leq b_n c_2(X)\, c_1(X)^{n-2}.$$

Thm 3 (KM type ineq for $\mathbb{Q}$-Fano. L-Liu 23). X be a canonical $\mathbb{Q}$-Fano var of dim n. smooth in codim 2. then.

Thm 3 (KM type ineq for $\mathbb{Q}$-Fano. L-Liu 23). X be a canonical $\mathbb{Q}$-F. var of dim n. smooth in codim 2. then.

$$c_1(X)^n < 4\, c_2(X)\, c_1(X)^{n-2}.$$

Moreover, in case $n=3$. then

$$c_1(X)^3 \leqslant \frac{25}{8}\, c_2(X)\, c_1(X)$$

§2. Background.

1). Bogomolov-Gieseker ineq. (79') (BG ineq)

E : torsion free sheaf of $rk\, r \geqslant 2$. on proj mfd X.

If E is $\boxed{\text{semi-stable}}$ w.r.t H. H ample on X. then

$$c_1(E)^2 H^{n-2} \leqslant \boxed{\frac{2r}{r-1}}\, c_2(E)\, H^{n-2}.$$

2). $E = T_X$. (or Ω_X^1.) (without semi-stability)

2.1). when K_X is nef. (abundance $\Rightarrow$ semiample).

a) Miyaoka-Yau ineq.

① (Miyaoka. Yau. 77') X. $\boxed{\text{smooth}}$ + K_X ample.

$$(K_X)^n = c_1(X)^2 K_X^{n-2} \leqslant \frac{2(n+1)}{n}\, c_2(X)\, K_X^{n-2} \overset{BG}{\leqslant} \left(\boxed{\frac{2n}{n-1}} \cdots\right)$$

② (Greb-Kebekus-Peternell-Taji 19') X. $\boxed{\text{klt}}$ (smooth in codim 2) & K_X nef + big.

$$(K_X)^n \leqslant \frac{2(n+1)}{n}\, c_2(X)\, K_X^{n-2}. \quad (\hat{c}_2)$$

b). Miyaoka ineq.

· (Miyaoka. 87') X terminal. & K_X nef.

$c_2(X)$ is pseff. i.e. $c_2(X)\cdot H_1 \cdots H_{n-2} \geqslant 0$ $\forall H_i$ nef (ample)

2.2) when $-K_X$ is nef.

b') Miyaoka type ineq.

① (Peternell 12') X smooth & $-K_X$ semiample. (Rmk. $-K_X$ nef $\not\Rightarrow$ semiample)

$c_2(X)$ is pseff.

② (Ou 23') X lc + smooth in codim 2. & $-K_X$ nef.

$c_2(X)$ is [...].

② [Ou. 23') X lc + smooth in codim 2. & $-K_X$ nef.

$c_2(X)$ is pseff.

a') Kawama-Miyaoka type ineq.

smooth {

① (Peternell. 12') X smooth. & $-K_X$ ample.

$$c_1(X)^n \le b_n c_2(X) c_1(X)^{n-2}.$$

constant depends on n.

Thm 1

↑

$c_2(X) c_1(X)^{n-2} > 0$

$\ne$ $\ge$ ten

② [Jie Liu 19') X smooth + $\rho(X)=1$. & $-K_X$ ample.

$$c_1(X)^n \le 4\, c_2(X) c_1(X)^{n-2}.$$

$f(q) \le 4$.

more precise depending on q: Fano index.

Sing. {

③ (Kawamata 92') X terminal $\mathbb{Q}$-Fano 3-fold.

$$c_1(X)^3 \le b_3 c_2(X) c_1(X)$$

constant.

④ (Iwai-Jiang-L. 23'). X. (ε-lc, >0), smooth in codim 2. & $-K_X$ is nef + big

$$c_1(X)^n \le b_{n,\varepsilon}\, c_2(X) c_1(X)^{n-2}.$$

constant depends only n. ε.

⑤ (L-Liu 23'). X. terminal $\mathbb{Q}$-Fano.

$$c_1(X)^n \le 4\, c_2(X) c_1(X)^{n-2}$$

§3. sketch of Pfs.

Thm 1. → [Ou 23'] c_2 pseff. key technique. ⇒ $c_2(X) c_1(X)^{n-2} \ge 0$

X weak Fano. (klt ~~+~~ smooth in codim 2)

Thm 2 → Thm 1 + BAG thm.

Thm 3 → r := rank of the $\boxed{\text{maximal destabling sheaf}}$ of T_X

r ≥ 2. ⇒ { Langer's ineq / Iwai-L. Jiang } ⇒ due back to Miyaoka 87'.

r = 1. ⇒ Fano foliation. → due back to Miyaoka [93'].

$\boxed{-K_{X/C}}$ for $f: X \to C$ smooth morp from X to curve C. can NOT be

(1) ...

$F = \mathbb{P}^1 \quad U \to X$, $U \downarrow$ klt, T

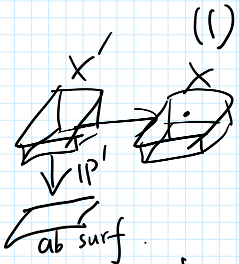
$\boxed{-K_{X/C}}$ for $f: X^s \to C$ smooth morp from X to curve C, can NOT be ample.

generalize. $X \to C$ klt case $\to$ Fano foliation.

§4. On terminal ($\mathbb{Q}$-Fano) 3-folds.

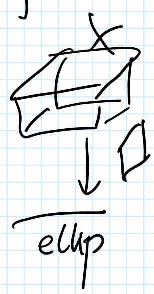
Aim: Classification!

(1). $\boxed{-K_X \text{ nef}}$. Then $c_2(X)c_1(X) = 0$ iff one of.



① . X admits a quasi-étale cover by a $\mathbb{P}^1$-bundle over an abelian surf.

② . X is smooth admitting a locally trivial fibration over an elliptic curve where fiber are rationally connected



$(r.b)$ $(1,-1,b)$ (local index

③ . X is RC. $(-K_X)^3 = 0$. and the set $\boxed{R_X}$ (Reid's basket) is only one of 8 types. $\to (2, \dots, 2)$ 16 8 (2^{16})

application of thm 1.

(2). $\mathbb{Q}$-Fano 3-folds $\quad -K_X$ ample $+ \rho(X) = 1 + \mathbb{Q}$-factorial.

Rmk 1. smooth case. 105? 106? (Iskovskih. Shokurov. Fujita. Mori. Mukai ...)

- sing case. (Partial result. Mukai. Sano. Campana. Fletcher ...)

Index (Suzuki Prokhorov ...)

$$qQ(X) = \max\{q \mid -K_X \sim_{\mathbb{Q}} qA \quad A \in Cl(X)\}$$

$$qW(X) = \quad \cdots \sim qB. \quad B \in \quad \}$$

$$b_X = \frac{c_1^3}{c_2 c_1}$$

$$qW(X) \mid qQ(X) \in \{1 \cdots 9, 11, 13, 17, 19\}$$

$? \leq b_X \leq ?$ $[\quad]$

- Graded Ring Database. (GRDB. Brown. Kasprzyk).

$\boxed{\text{Hilbert series } 50000 \, f.}$ $\leftarrow$

Rmk 2. two key tools. $\begin{cases} \text{Reid's orbifold RR formula.} \\ \text{Kawamata ineq in dim 3} \end{cases}$

Thm (-. Liu 23) X. terminal $\mathbb{Q}$-Fano 3-fold. then

$qQ = qW = 5$

Thm (-, Liu 23) X. terminal $\mathbb{Q}$-Fano 3-fold. then

$$c_1(X)^3 \le \frac{25}{8} c_2(X) c_1(X).$$

"=" holds iff $R_X = \{3, 7, 7\}$, $\cancel{X}$ $qQ = qW = 5$

Rmk: rule out lots of possibilities of Hil series. (20%?)

- Liu and I (possibly Prokhorov?) improve to.

$$c_1(X)^3 < 3 c_2(X) c_1(X).$$

- $$c_1(X)^3 \le 3 c_2(X) c_1(X)$$

"=" holds iff. $R_X = \{7, 13\}$. $qQ = qW = 4$. X. □