Plan :

(I) Motivation (matching objects in homological mirror symmetry)

(II) Invertible Polynomials (Kruezer-Skarke classification, Milnor #'s, cleaves)

(III) Symmetry Groups

(IV) Matrix factorizations (equiv. matrix factorization cat. BFK-decomposition, lit. rev.)

(V) Main result (Gorenstein case, toy example, exc. objs)

Non-compact mirror symmetry [Kontsevich, Orlov]

Fukaya-Seidel cat

$(\mathbb{C}^n, w^T, 1) + (\mathbb{C}^n, w, \Gamma_w)$ are mirror if proposed mirror of Berglund–Hübsch

matrix factorization cat $\updownarrow$ Orlov

Γ_w-equiv singular sheaves on $w^{-1}(0)$

intersection pattern of vanishing cycles

$$FS(\mathbb{C}^n, w^T) \overset{?}{\cong} mf(\mathbb{C}^n, w, \Gamma_w)$$

A_∞-cat

obj $\supset \{L_i\}$ Seidel

L_i = Lefschetz thimbols

$\{m_j\} \subset$ obj

$FS(\mathbb{C}^n, w^T) \overset{\text{if } \{L_i\} \text{ generate}}{\cong} End(\oplus L_i)\text{-dg mod}$

$mf(\mathbb{C}^n, w, \Gamma_w) \overset{\text{if } \{m_j\} \text{ gen.}}{\cong} End(\oplus m_j)\text{-dg mod}$ Dyckerhoff

$$End(\oplus L_i)\text{-dg mod} \overset{?}{\cong} End(\oplus m_j)\text{-dg mod}$$

A_∞-alg

$End(\oplus L_i)\text{-dg mod} \overset{\text{if formal}}{\cong} H_*(End(\oplus L_i))\text{-mod}$

$\star$ $End(\oplus m_j)\text{-dg mod} \overset{\text{formal}}{\cong} H_*(End(\oplus m_j))\text{-mod}$

Ballard–Favero–Katzarkov

Polishchuk–Vaintrob

m_j = (sky-scrapper at the origin twisted by $g \in \Gamma_w/\mathbb{G}_m$)

$$H_*(End(\oplus L_i))\text{-mod} \overset{?}{\cong}_{\text{alg}} H_*(End(\oplus m_j))\text{-mod}$$

What conditions can we put on $\{m_j\}$ to get $\star$?

Defn: $\{m_j\}$ is a strong exc. collection $(\Rightarrow \oplus m_j$ is tilting$)$

$$\bigoplus_{n \in \mathbb{Z}} Hom(m_i, m_j[n]) = \begin{cases} Hom(m_i, m_j) & \text{if } i < j \\ \mathbb{C} \cdot Id_{m_i} & \text{if } i = j \\ 0 & \text{if } i > j \end{cases}$$

Main result [Favero – K – Kelly]

$mf(\mathbb{C}^n, w, \Gamma_w)$ has an full (i.e. it generates) exc. collection for w an invertible poly.

(V) (IV) (II) (III)

but not necessarily strong

strong in the Gorenstein case

Notation: fec := full exc collection

(II) Invertible Polynomials

Defn: $w \in \mathbb{C}[x_1, x_n]$ is invertible if $w = \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ij}}$

and (1) $A = [a_{ij}] \in mat_{n \times n}(\mathbb{Q})$ is invertible

(2) w is quasi-homogeneous

i.e. $\exists\, \underbrace{q_1, q_n}_{\text{weights}}, \underbrace{d}_{\text{degree}}$ s.t. $\sum_{j=1}^{n} a_{ij} q_j = d$

(i.e. if $\deg(x_j) = q_j$ then w is weighted homogeneous of deg. d)

(3) w is quasi-smooth i.e. $sing(w) = \{0\}$

Ex and non-ex

(a) $w(x,y) = x^2 + xy + y^2$ not sum of 2-monomials

(b) $w(x,y) = xy + \underbrace{x^2y^3}_{\text{term dominates}}$ not quasi-homo

(c) $w(x,y) = x^2y + x^3 = x^2(y+x)$ $sing(w) = \{x=0\}$ not q.s.

(d) $w(x,y) = x^2y + y^5$ invertible !

(1) $\det\left(\begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}\right) = 10 \neq 0$

(2) $q_1 = 2,\ q_2 = 1,\ d = 5$

(3) check : $V\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right) = \{0\}$

Kreuzer - Skarke classification

Defn: Let $f \in \mathbb{C}[x_1, \dots, x_n]$ $\quad g \in \mathbb{C}[x_1, \dots, x_m]$

The Thom-Sebastiani sum of f and g is $f \boxplus g\,(x_1, \dots, x_{n+m}) = f(x_1, \dots, x_n) + g(x_{n+1}, \dots, x_{n+m})$

Thm [Kreuzer-Skarke]

Let w be invertible. Then w is a Thom-Seb. sum of "atomic" polynomials:

(1) Fermat $\quad x^a \qquad \bullet \qquad\qquad 1 \to 2 \to \cdots \to n \qquad a > 1$

(2) Chain $\quad x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} \qquad a_i > 1$

(3) Loop $\quad$ Chain $+ x_n^{a_n} x_1 \qquad 1 \to \cdot \to \cdots \to n$

Rem: Given $w \rightsquigarrow$ a directed graph with vertex i for each variable x_i

an arrow $i \to j$ if w has a term of the form $x_i^{a_i} x_j$

Example: ADE polynomials are invertible

e.g. $D_n = \underbrace{x^2 y + y^{n-1}}_{\text{length 1 chain}} + \underbrace{z^2}_{\text{Fermat}}$

Defn: The Milnor # of w is the $\dim \mathbb{C}[x_1, \dots, x_n] / \left(\frac{\partial w}{\partial x_i}\right)$

$=$ # of repeated roots

$=$ # of roots of a morsification

Defn: • $f, g \in \mathbb{C}[x_1,, x_n]$ are related by an elementary cleave $\underset{\wedge}{W}$ if $W|_{\{x_{n+1}=1\}} = f$ and $W|_{\{x_n=1\}} = g$ $\mathbb{C}[x_1,, x_{n+1}]$

for this talk

• f, g are related by a cleave if $\exists f_1,, f_m$ and $\exists W_i$ an elem. cleave relating f_i and f_{i+1} $\forall i$ with $f = f_1$ and $g = f_m$

Prop: The n-loop + n-chain are related by the elementary cleave $n+1 \to 1 \to \cdots \to n$

$$W = x_1^{a_1} x_2 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1 x_{n+1}^b$$

and both are related by a cleave to a sum of Fermats

Pf | $W_n' := x_1^{a_1} x_2 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_{n+1}^b$

is an elem. cleave from the n-chain to the n-1 chain + Fermat $1 \to 2 \to \cdots \to n$

$1 \to 2 \to \cdots \to n-1$

so W_n', $W_{n-1}' \boxplus id_{x_n}$, ..., $W_2' \boxplus id_{x_3,..x_n}$

is a sequence of elem. cleaves from chain to a sum of Fermats □

Idea of the proof that $mf(\mathbb{C}^n, w, \Gamma_w)$ has an f.e.c

(1) Reduce statement to w = chain, loop, Fermat

(2) Show if w, w' are related by a cleave and w has fec $\mathcal{E}_w$ then w' has fec $\mathcal{E}_{w'}$ } reduces to Fermat

an equivalence if $\mu(w) = \mu(w')$

(3) $mf(\mathbb{C}^n, x^a, \Gamma_{x^a})$ has fec $\mathcal{E}_{x^a} := \{ \mathbb{C} \xrightarrow{x^i} \mathbb{C} \xrightarrow{x^{a-i}} \mathbb{C} \}_{i=1,,a-1}$

(III) Symmetry groups let $w \in \mathbb{C}[x_1,..,x_n]$ $w' \in \mathbb{C}[x_1,..,x_m]$

$$\Gamma_w := \{ (\lambda_1, .., \lambda_{n+1}) \in \mathbb{G}_m^{n+1} \mid w(\lambda_1 x_1, .., \lambda_n x_n) = \lambda_{n+1} w(x_1, .., x_n) \}$$

Rem: If w is quasi-homo then $\exists\, q_i, d$ s.t.

$$w(\lambda^{q_1} x_1, .., \lambda^{q_n} x_n) = \lambda^d w(x_1, .., x_n)$$

so

$$\mathbb{G}_m \xrightarrow{\phi} \Gamma_w$$
$$\lambda \longmapsto (\lambda^{q_1}, .., \lambda^{q_n}, \lambda^d)$$

in fact coker(ϕ) is torsion for w inv.

Γ_w acts on $\mathbb{C}^n$ by projecting to first n factors

$$\begin{tikzcd} \Gamma_{w \boxplus w'} & \Gamma_w \times \Gamma_{w'} \arrow[l] & \mathbb{G}_m \arrow[l] \\ \mathbb{G}_m^{n+m+1} \arrow[u, phantom, "\cap" rotate=180] & \mathbb{G}_m^{n+m+2} \arrow[u, phantom, "\cap" rotate=180] & \end{tikzcd}$$

note: λ_{n+1} and λ'_{m+1} could be different

so glue them

so

$$\Gamma_{w \boxplus w'} \cong \Gamma_w \times_{\mathbb{G}_m} \Gamma_{w'}$$

Ex 1: $w = x^q$ $\quad \Gamma_w = \{ (\lambda_1, \lambda_2) \mid (\lambda_1 x)^q = \lambda_2 x^q \}$

$$= \{ (\lambda_1, \lambda_1^q) \}$$

$$\cong \mathbb{G}_m$$

Ex 2: $w = x^{a_1} + y^{a_2}$ $\quad \Gamma_w = \mathbb{G}_m \overset{a_1, a_2}{\underset{\mathbb{G}_m}{\times}} \mathbb{G}_m$

$$\cong \mathbb{G}_m \times \mathbb{Z}/\gcd(a_1, a_2)\mathbb{Z}$$

e.g. $x^2 + y^2$ $\quad \Gamma_w = \langle \underset{\mathbb{G}_m}{\underset{\shortparallel}{(\lambda, \lambda, \lambda^2)}}, \underset{\mathbb{Z}/2\mathbb{Z}}{\underset{\shortparallel}{(-1, 1, 1)}} \rangle$

note: $(1, -1, 1) = (-1, 1, 1) \cdot (-1, -1, 1)$

Ex 3: $w = x^2 y + y^2 z + z^2 x$

primitive 3rd root of unity

$$\Gamma_w = \langle (\lambda, \lambda, \lambda, \lambda^3), (\zeta, \zeta^{-1}, 1, \zeta) \rangle$$

$$\cong \mathbb{G}_m \times \mathbb{Z}/3\mathbb{Z}$$

Upshot: Γ_w is computable for w invertible

Rem:

$$\Gamma_f \xrightarrow{\lambda_{n+1} = 1} \Gamma_w \xleftarrow{\lambda_n = 1} \Gamma_g$$

Γ_w — elem. cleave from f to g

(IV) Equiv. matrix factorization category

<u>Defn:</u> A Γ_w-equiv. matrix factorization for $w \in \mathbb{C}[x_1, x_n]$ is

$$P_0 \xrightarrow{d_0} P_1(\chi) \xrightarrow{d_1} P_0(\deg(w))$$

d_0, d_1 are Γ_w-inv.

P_0, P_1 are ~~projective~~ free Γ_w-equiv. $\mathbb{C}[x_1,, x_n]$-modules of dim n_0, n_1

so
$$d_1 \circ d_0 = \begin{bmatrix} w & & \\ & \ddots & \\ & & w \end{bmatrix} \in \mathrm{Mat}_{n_0 \times n_0}(\mathbb{C}[x_1, x_n])$$

$$d_0 \circ d_1 = w\, Id_{n_1 \times n_1}$$

<u>Defn:</u> $mf(\mathbb{C}^n, w, \Gamma_w)$ = cat w/ obj above
morphisms are (homotopy classes of) chain maps

Ex: For $w = x^a$ $\Gamma_w = G_m$-equiv $\Leftrightarrow \hat{\Gamma}_w \cong \mathbb{Z}$-graded

$$\begin{tikzcd}
E_1[1] & \bullet\ 1 \arrow[d] & \mathbb{C}[x] \arrow[r, "x"] \arrow[d, "1"] & \mathbb{C}[x][1] \arrow[r, "x^{a-1}"] \arrow[d, "x"] & \mathbb{C}[x][a] \arrow[d, "1"] & =: E_1 \\
 & \bullet\ 2 \arrow[d] \arrow[u, "\times"] \arrow[ul, bend left] & \mathbb{C}[x] \arrow[r, "x^2"] \arrow[d] \arrow[u, bend left, "x"] & \mathbb{C}[x][2] \arrow[r, "x^{a-2}"] \arrow[d] \arrow[u, bend left, "1"] & \mathbb{C}[x][a] \arrow[d] \arrow[u, bend left, "x"] & =: E_2 \\
\text{Serre functor } [1] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & \bullet\ a-1 & \mathbb{C}[x] \arrow[r, "x^{a-1}"] & \mathbb{C}[x][a-1] \arrow[r] & \mathbb{C}[x][a] & =: E_{a-1}
\end{tikzcd}$$

$$mf(\mathbb{C}, x^a, \Gamma_{x^a}) \simeq D^b(\mathrm{Reps}(\bullet \to \bullet \to \cdots \to \bullet))$$

Thm [Ballard–Favero–Katzarkov, Polishchuk–Vaintrob] — specialized + simplified

A cat. of kernels of equiv mf II

MF and CFT

Let w, w' be invertible.

$$mf(\mathbb{C}^{n+m}, w \boxplus w', \Gamma_{w \boxplus w'})$$
$$\cong$$
$$mf(\mathbb{C}^n, w, \Gamma_w) \otimes mf(\mathbb{C}^m, w', \Gamma_{w'})$$

derived cat of dg modules over the tensor product

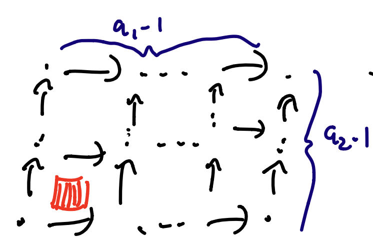
Cor: $\mathcal{E}_{w \boxplus w'} = \mathcal{E}_w \times \mathcal{E}_{w'}$ is a fec if $\mathcal{E}_w, \mathcal{E}_{w'}$ are fec

i.e. order lexicographically

e.g. $mf(\mathbb{C}^2, x_1^{a_1} + x_2^{a_2}, \Gamma_{x_1^{a_1} + x_2^{a_2}})$

$$\cong$$

D^b Reps (



)

squares commute

Literature review

$mf(\mathbb{C}^n, w, \Gamma_w)$ has a full strong exceptional collection for

- w Brieskorn – Pham (ie. $w(x_1, x_n) = x_1^{a_1} + \dots + x_n^{a_n}$) by Futaki-Ueda
- w chain by Hirano-Ouchi and indepenedently Aramaki-Takahashi
- w invertible in 2-variables by Habermann - Smith
- w invertible in 3-variables by Kravets

If $w = \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ij}}$ then define $w^T := \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ji}}$

$$mf(\mathbb{C}^n, w, \Gamma_w) \overset{A_\infty\text{-cat}}{\simeq} FS(\mathbb{C}^n, w^T)$$

for

- w Brieskorn – Pham by Futaki-Ueda
- w ADE by Takahashi
- w invertible in 2-variables by Habermann - Smith

(V) Exceptional collections for $mf(\mathbb{C}^n, w, T_w)$

Let W be an elem. cleave from w, w/ W invertible

$$\begin{tikzcd}[column sep=small]
& mf(\mathbb{C}^{n+1}, W, T_W) \arrow[dl, shift right, "\{x_{n+1}=1\}"'] \arrow[dr, shift left, "\{x_n=1\}"] & \\
mf(\mathbb{C}^n, w, T_w) \arrow[ur, shift right, "\Phi_-"'] & \text{fully faithful functors} & mf(\mathbb{C}^n, w', T_{w'}) \arrow[ul, shift left, "\Phi_+"]
\end{tikzcd}$$

Roughly, try to match up $\mathrm{im}(\Phi_-)$ and $\mathrm{im}(\Phi_+)$ in $mf(\mathbb{C}^{n+1}, W, T_W)$

But, their "sizes" are different in general

the length of $\mathcal{E}_w$ is (a posteriori) $= \mu(w)$

Thm [Ballard–Favero–Katzarkov, 2019; Favero–K–Kelly, 2020] Variation of GIT

Assume w, w' are rel. by an elem. cleave and $\mu(w') \leq \mu(w)$.

$mf(\mathbb{C}^n, w, T_w)$ has a semi-orth. decomposition

$$\langle\, mf(\mathbb{C}^n, w', T_{w'}),\ E_1, \ldots, E_{\mu(w)-\mu(w')} \,\rangle$$

In particular, if $\mu(w') = \mu(w)$ the categories are equivalent

And # of exc. obj in any fec in $mf(\mathbb{C}^n, w, T_w)$ is $\mu(w)$

Cor: $mf(\mathbb{C}^n, w, T_w)$ has a fec ~ strong in the equiv. case (Gorenstein)

Toy Example: $W = x^r y$

$$\swarrow_{\{x=1\}} \qquad \searrow_{\{y=1\}}$$

$w = y$ $\qquad$ $w' = x^r$

$\mu(w) = 0$ smooth $\qquad$ $\mu(w') = r-1$

$$mf(\mathbb{C}, y, \mathbb{G}_m) \qquad mf(\mathbb{C}, x^r, \mathbb{G}_m)$$

$$\| \qquad\qquad \|$$

$$0 \qquad\qquad \langle E_1', \dots, E_{r-1}' \rangle$$

Main result gives a different perspective on these $r-1$ obj as $r-1$ copies of

$$\underbrace{mf\big(\underset{\underset{\text{origin}}{\|}}{\text{fixed locus } T_w \circlearrowright \mathbb{C}^2}, 0, \underset{\underset{\mathbb{G}_m}{\cong}}{\operatorname{coker}(\lambda)}\big)}_{\underset{\underset{\underset{\mathrm{Vect}_{\mathbb{C}} = \langle \mathbb{C} \rangle}{\cong}}{D^b\mathrm{Coh}(pt)}}{\cong}}$$

where

$$\lambda : \mathbb{G}_m \to T_W \qquad t \mapsto (t^{-r}, t, 1)$$

$$\|$$

$$\ker(T_W \xrightarrow{\pi_3} \mathbb{G}_m) \quad \text{i.e.} \quad \operatorname{coker}(\lambda) \cong \pi_3(T_W) \cong \mathbb{G}_m$$

Thank you for listening!