Explicit boundedness of canonical Fano 3-folds

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July 8, 2022

We work over $\mathbb{C}$.

Definition

A normal projective variety X is called

- $\mathbb{Q}$-**Fano** if $-K_X$ is ample;
- **weak** $\mathbb{Q}$-**Fano** if $-K_X$ is nef and big.

According to the Minimal Model Program, (weak) $\mathbb{Q}$-Fano varieties form a fundamental class in birational geometry.

Examples of smooth Fano varieties

- $\mathbb{P}^n$;
- smooth hypersurfaces in $\mathbb{P}^n$ of degree $\leq n$;
- In dimension 1, $\mathbb{P}^1$.
- In dimension 2, $\mathbb{P}^1 \times \mathbb{P}^1$ or blowing up $\mathbb{P}^2$ at ≤ 8 general points.

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- In dimension 2, $\mathbb{P}^1 \times \mathbb{P}^1$ or blowing up $\mathbb{P}^2$ at ≤ 8 general points.
- In dimension 3, exactly 105 deformation families (Iskovskikh, Mori–Mukai)

In general it is every hard to classify all $\mathbb{Q}$-Fano varieties (in higher dimensions or with worse singularities).
In this talk we mainly consider (weak) $\mathbb{Q}$-Fano 3-folds with terminal/canonical singularities.

Definition

Let X be a normal variety such that K_X is $\mathbb{Q}$-Cartier.
Let $f : Y \to X$ be a resolution. Write $K_Y = f^*K_X + \sum_i a_i E_i$.

- X is **terminal** if $a_i > 0$ for all i.
- X is **canonical** if $a_i \geq 0$ for all i.

- In dimension 2, terminal $\Longleftrightarrow$ smooth; canonical $\Longleftrightarrow$ Du Val.
- Introduced by Reid, appearing naturally in MMP.
- Terminal singularities in dimension 3 are classified by Mori.

Theorem (Kawamata, Kollár–Miyaoka–Mori–Takagi, Birkar)

Fix $d \in \mathbb{Z}_{>0}$. The set of all canonical weak $\mathbb{Q}$-Fano varieties of dimension d is a bounded family (i.e., there are only finitely many deformation classes).

Goal

- Study explicit boundedness of invariants;
- Classify extremal cases.

We are mainly interested in the following invariants:

- anti-canonical volume $(-K_X)^3$.
- pluri-anti-canonical systems $|-mK_X|$;

Question

Let X be a canonical weak $\mathbb{Q}$-Fano 3-fold. What is the lower/upper bound of $(-K_X)^3$?

For the lower bound:

Theorem (Chen–Chen 08)

Let X be a canonical weak $\mathbb{Q}$-Fano 3-fold. Then $(-K_X)^3 \geq \frac{1}{330}$ (optimal). Moreover, if $(-K_X)^3 = \frac{1}{330}$, then X has the same Hilbert series (same $h^0(X, -mK_X)$) as

$$X_{66} \subset \mathbb{P}(1, 5, 6, 22, 33).$$

Question

How to characterize the extremal case when $(-K_X)^3 = \frac{1}{330}$? Is it always a $\mathbb{Q}$-Gorenstein deformation of X_{66}?

We get a partial answer to this question in a smaller category. Note that a general X_{66} is a $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano 3-fold with $\rho(X) = 1$.

Theorem (J. 21 + J. 22 (new result in this week))

Let X be a $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano 3-fold with $\rho(X) = 1$ and $(-K_X)^3 = \frac{1}{330}$. Then X is a weighted hypersurface of degree 66 in $\mathbb{P}(1, 5, 6, 22, 33)$.

In fact, we can get the same result for 12 weighted hypersurfaces of the form $X_{6d} \subset \mathbb{P}(1, a, b, 2d, 3d)$ in Iano-Fletcher's list.

Theorem (J. 22)

Let X be a $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano 3-fold with $\rho(X)=1$. If X has the same Hilbert series as some $X_{6d}\subset\mathbb{P}(1,a,b,2d,3d)$ in Iano-Fletcher's list, then X itself is a weighted hypersurface of the same type.

Step 1 take general $f_m \in H^0(X,-mK_X)$ for $m=1,a,b,2d,3d$, define

$$\Phi : X \dashrightarrow \mathbb{P}(1,a,b,2d,3d);$$
$$P \mapsto [f_1(P):f_a(P):f_b(P):f_{2d}(P):f_{3d}(P)]$$

Step 2 Show that Φ defines a birational map onto its image Y, and Y is a weighted hypersurface of degree $6d$; (Hint: by results of [Chen–J. 16], $|-2dK_X|$ defines a generically finite map of degree 2, and $|-3dK_X|$ defines a birational map)

Step 3 Show that $X \simeq Y$ by comparing Hilbert series.

For the upper bound:

- $(-K_X)^3 \leq 6^3 \cdot (24!)^2$ if X is a terminal weak $\mathbb{Q}$-Fano 3-fold whose anti-canonical map is small [KMMT 00]; (Bend and break)
- $(-K_X)^3 \leq 64$ if X is a Gorenstein terminal $\mathbb{Q}$-Fano 3-fold, " $=$ " iff $X \simeq \mathbb{P}^3$ [Namikawa 97]; (deformation)
- $(-K_X)^3 \leq 72$ if X is a Gorenstein canonical $\mathbb{Q}$-Fano 3-fold, " $=$ " iff $X \simeq \mathbb{P}(1,1,1,3)$ or $\mathbb{P}(1,1,4,6)$ [Prokhorov 05]; (MMP)
- $(-K_X)^3 \leq \frac{125}{2}$ if X is a non-Gorenstein $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano 3-fold with $\rho(X) = 1$, " $=$ " iff $X \simeq \mathbb{P}(1,1,1,2)$ [Prokhorov 07]; (MMP, Sarkisov links, Riemman–Roch)
- $(-K_X)^3 \leq 72$ if X is a $\mathbb{Q}$-factorial terminal weak $\mathbb{Q}$-Fano 3-fold with $\rho(X) = 2$ except in one case (≤ 81) [Lai 21]. (MMP, Sarkisov links)

In general, it is expected that for a canonical weak $\mathbb{Q}$-Fano 3-fold X, $(-K_X)^3 \leq 72$, but even an explicit bound is not yet established.

Theorem (J.–Zou 21)

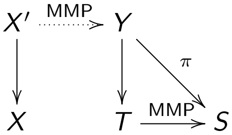
Let X be a canonical weak $\mathbb{Q}$-Fano 3-fold. Then $(-K_X)^3 \leq 324$.

Corollary

Let X be a canonical weak $\mathbb{Q}$-Fano 3-fold. Then $h^0(X, -K_X) \leq 164$.

- In my thesis, I gave a general strategy on bounding anti-canonical volumes of $\mathbb{Q}$-Fano varieties with prescribed singularities. For example, I showed that for a weak $\mathbb{Q}$-Fano 3-fold with ϵ-lc singularities, there exists a number $M(\epsilon)$ such that $(-K_X)^3 \leq M(\epsilon)$. (canonical=1-lc).
- The problem is to make the above strategy as explicit as possible.
- In fact, our method gives an explicit bound for $M(\epsilon)$.

The reduction step:



Proposition (Reduction to a birational model)

Let X be a canonical weak $\mathbb{Q}$-Fano 3-fold. Then X is birational to a normal projective 3-fold Y satisfying the following:

- *Y is $\mathbb{Q}$-factorial terminal;*
- $(-K_X)^3 \leq \mathrm{Vol}(Y, -K_Y) = \lim_{m\to\infty} \frac{h^0(Y,-mK_Y)}{m^3/6}$;
- *$|-nK_Y|$ is movable for sufficiently large and divisible n;*
- *for a general member $M \in |-nK_Y|$, $(Y, \frac{1}{n}M)$ is canonical;*
- *there exists a morphism $\pi : Y \to S$ with connected fibers where F is a general fiber of π, such that one of the following conditions holds:*
    - *S is a point and Y is a $\mathbb{Q}$-Fano 3-fold with $\rho(Y) = 1$;*
    - *$S = \mathbb{P}^1$ and F is a smooth weak del Pezzo surface;*
    - *S is a del Pezzo surface with Du Val singularities and $\rho(S) = 1$, and $F \simeq \mathbb{P}^1$.*

$$\begin{array}{ccccc} X' & \overset{\text{MMP}}{\dashrightarrow} & Y & & \\ \downarrow & & \downarrow & \overset{\pi}{\searrow} & \\ X & & T & \overset{\text{MMP}}{\longrightarrow} & S \end{array}$$

Proposition

- *If* $\dim S = 0$*, then* $\mathrm{Vol}(Y, -K_Y) \leq 64$ *[Prokhorov 07];*
- *If* $\dim S = 1$*, then* $\mathrm{Vol}(Y, -K_Y) \leq 324$*;*
- *If* $\dim S = 2$*, then* $\mathrm{Vol}(Y, -K_Y) \leq 312$*.*

Idea of the proof when $\dim S = 1$:

- Suppose $\mathrm{Vol}(Y, -K_Y) >> 0$, then we can use $-K_Y$ to construct singularities on F (connectedness lemma);
- Bound singularities of F (log canonical thresholds; α-invariants).

Proof of the case $\dim S = 1$:

Step 1. Assume to the contrary that $\mathrm{Vol}(Y, -K_Y) > 36K_F^2$, then we can find a rational number s such that

$$\mathrm{Vol}(Y, -K_Y) > 3sK_F^2 > 36K_F^2.$$

Then

$$\mathrm{Vol}(Y, -K_Y - sF) \geq \mathrm{Vol}(Y, -K_Y) - 3s\mathrm{Vol}(F, -K_F) > 0.$$

Hence there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_Y - sF$ on Y.

Step 2. Recall that $(Y, \frac{1}{n}M)$ is canonical, consider

$$\left(Y, \left(1 - \frac{2}{s}\right)\frac{1}{n}M + \frac{2}{s}D + F_1 + F_2\right)$$

where F_1, F_2 are general fibers of π, then the connectedness lemma shows that the nonklt locus of this pair is connected.

. Restricting to a general fiber F, we have

- $(F, \frac{1}{n}M|_F)$ is canonical;
- $(F, (1-\frac{2}{s})\frac{1}{n}M|_F + \frac{2}{s}D|_F)$ is not klt;
- F is a weak del Pezzo surface;
- $\frac{1}{n}M|_F \sim_{\mathbb{Q}} D|_F \sim_{\mathbb{Q}} -K_F$.

This is an lct-type problem.

Theorem (J.-Zou 21)

Under the above setting, $\frac{2}{s} \geq \frac{1}{6}$.

This contradicts

$$\mathrm{Vol}(Y, -K_Y) > 3sK_F^2 > 36K_F^2.$$

So $\mathrm{Vol}(Y, -K_Y) \leq 36K_F^2 \leq 324$.

Question

Let X be a canonical weak $\mathbb{Q}$-Fano 3-fold.

- *When $h^0(X, -mK_X) > 0$?*
- *When $h^0(X, -mK_X) \geq 2$?*
- *When does $|-mK_X|$ define a birational map?*

Theorem (Chen–Chen 08)

Let X be a canonical weak $\mathbb{Q}$-Fano 3-fold.

- *$h^0(X, -mK_X) > 0$ for any $m \geq 6$;*
- *$h^0(X, -8K_X) \geq 2$ (optimal).*

- So far there is no example with $h^0(X, -2K_X) = 0$;
- $X_{24,30} \subset \mathbb{P}(1, 8, 9, 10, 12, 15)$, $h^0(X, -7K_X) = 1$.

Theorem ([Chen–J. 16 & 21])

Let X be a canonical weak $\mathbb{Q}$-Fano 3-fold.

- $|-mK_X|$ *is birational for any* $m \geq 97$*;*
- $|-mK_X|$ *is birational for any* $m \geq 39$ *if X is a $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano 3-fold with* $\rho(X) = 1$*.*
- *X is birational to a terminal $\mathbb{Q}$-Fano 3-fold Y such that $|-mK_Y|$ is birational for any* $m \geq 52$*.*

- $X_{66} \subset \mathbb{P}(1, 5, 6, 22, 33)$, $|-33K_X|$ is birational but $|-32K_X|$ is not.

Theorem ([J.–Zou])

Let X be a canonical weak $\mathbb{Q}$-Fano 3-fold. Then $|-mK_X|$ is birational for any $m \geq 59$*.*

The main ingredient is to tell when $|-mK_X|$ is not composed with a pencil.

Theorem ([Chen–J. 16])

Let X be a canonical weak $\mathbb{Q}$-Fano 3-fold. If $h^0(X, -m_0K_X) \geq 2$ and $|-m_1K_X|$ is not a pencil, then $|-mK_X|$ is birational $m \geq 3(m_0 + m_1)$.

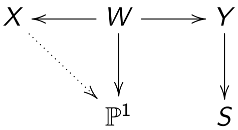
Theorem ([Chen–J. 16])

Let X be a canonical weak $\mathbb{Q}$-Fano 3-fold. If $|-mK_X|$ is pencil, then $h^0(-mK_X) \leq r_X(-K_X)^3m + 1$.

We can use Riemann–Roch to estimate $h^0(-mK_X)$ to find m breaking this inequality, but $r_X(-K_X)^3$ could be very large.

Theorem ([J.–Zou 22])

Let X be a canonical weak $\mathbb{Q}$-Fano 3-fold. If $|-mK_X|$ is pencil, then $h^0(-mK_X) \leq 12m + 1$.



Thank you for your attention!