Laurent Smoothing Turin Degenerations & Mirror Symmetry

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Laurent Mirror-Models

Playbill

Prehistoric Prelude

Meromorphic Madrigal

Minuet

March

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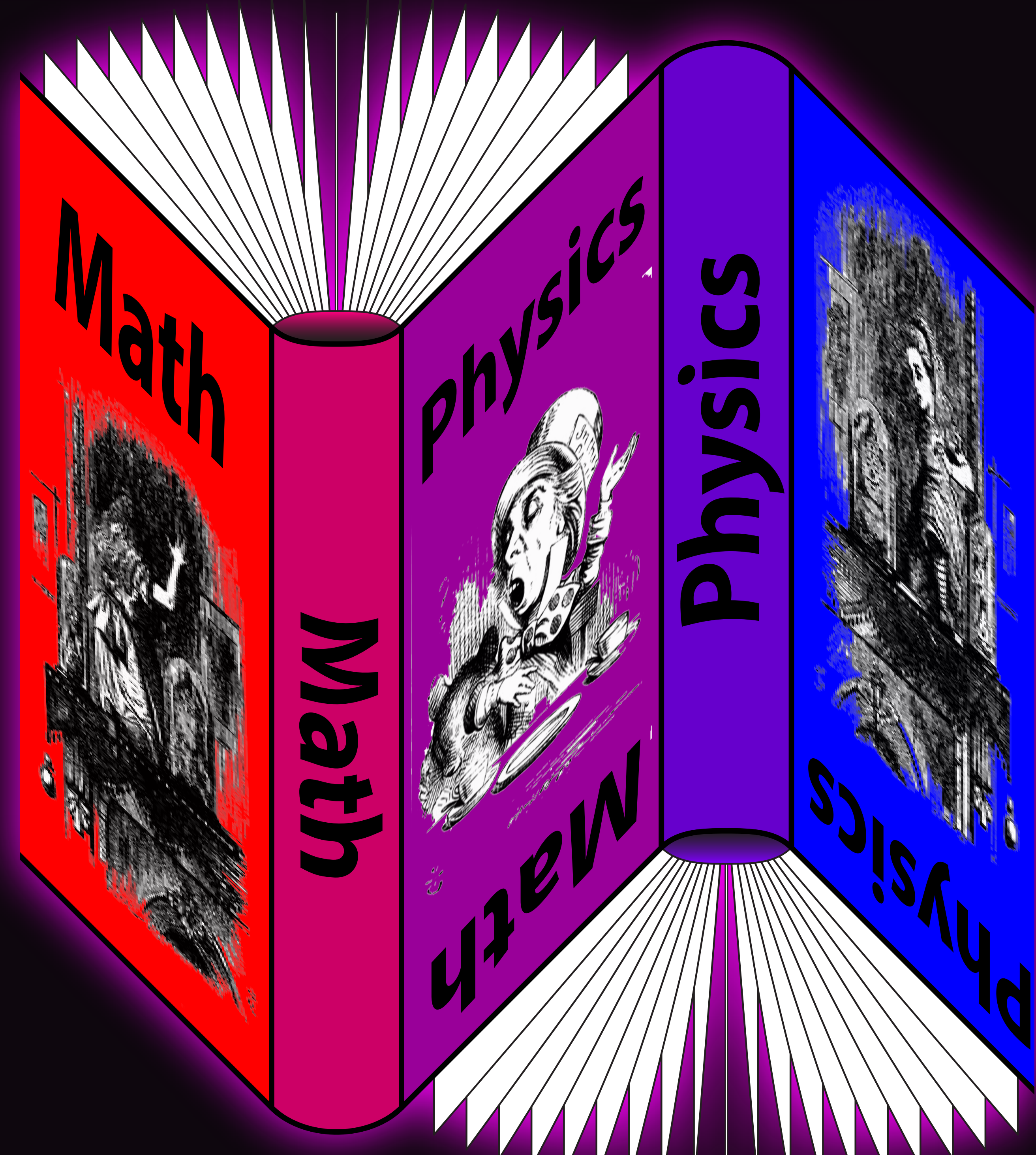
*Laurent-Toric Fugue**

Discriminant Divertimento

Mirror Motets

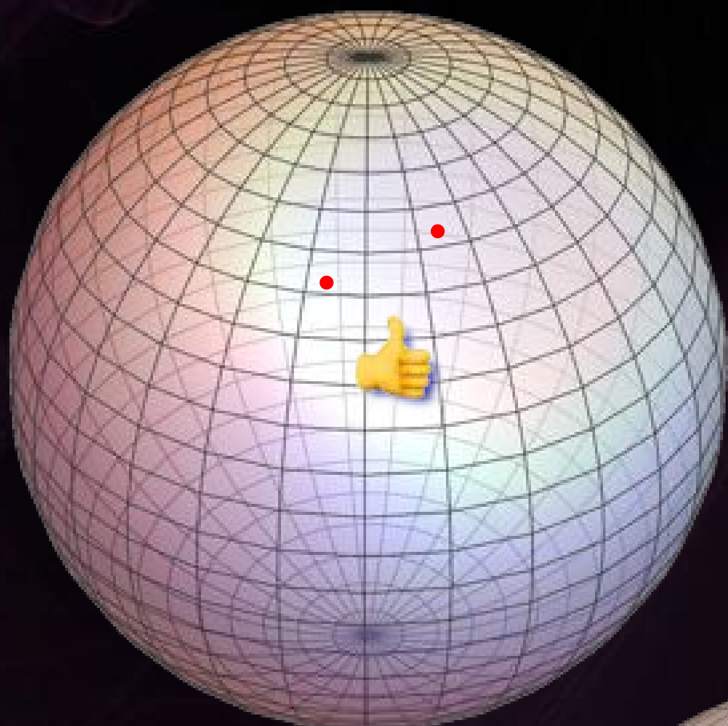
* "It doesn't matter what it's called,
...as long as it has substance."
— S.-T. Yau

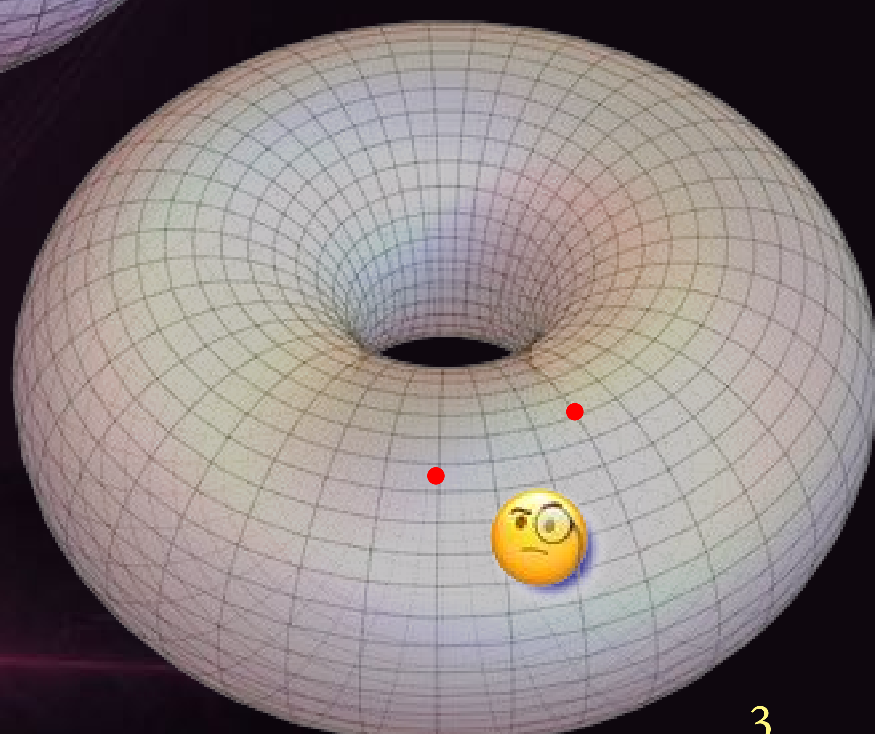


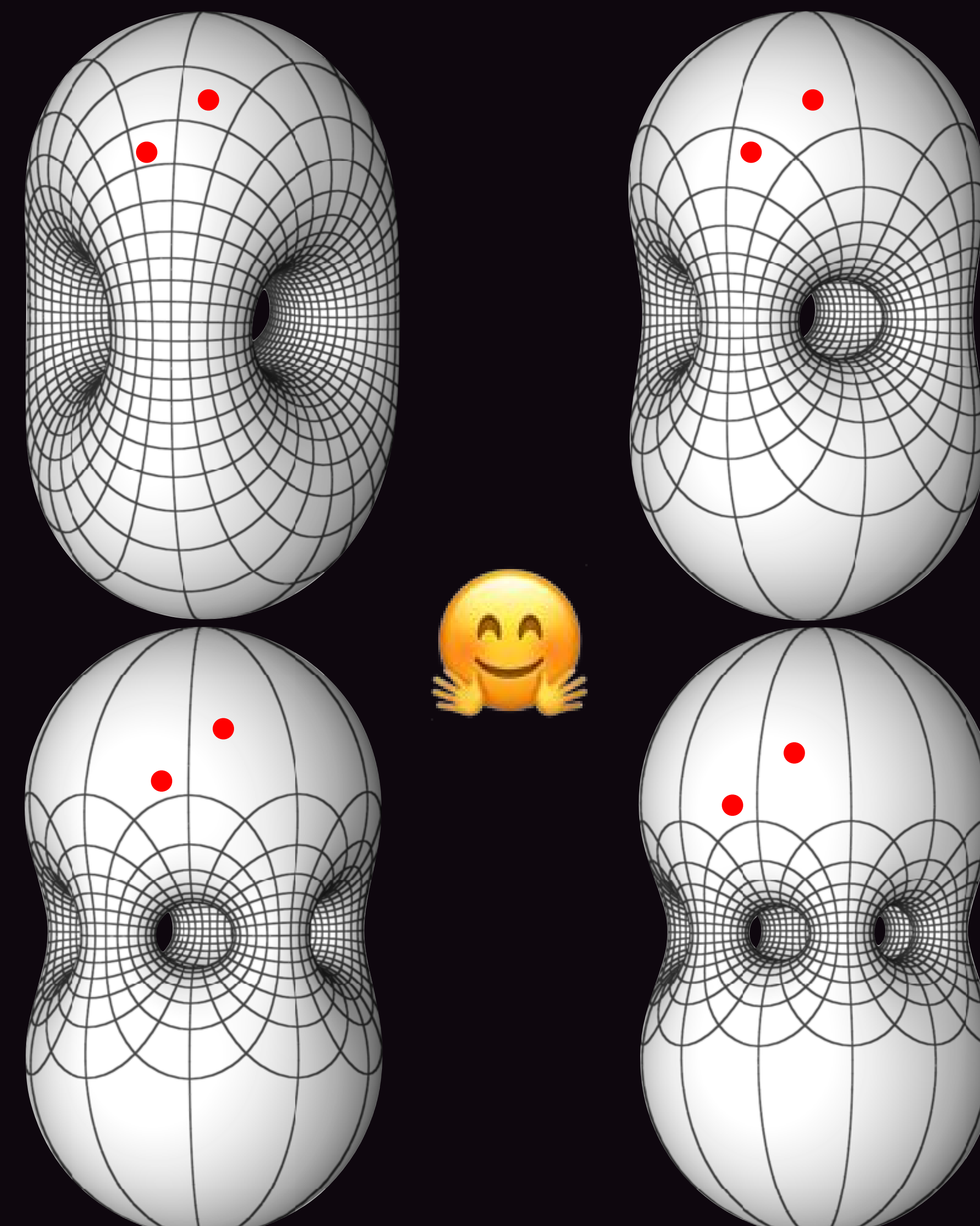
How Hard Can it Be?

Constructing CY ⊂ Some “Nice” Ambient Space

- Reduce to 0 dimensions: $\mathbb{P}^4[5] \to \mathbb{P}^3[4] \to \mathbb{P}^2[3] \to \mathbb{P}^1[2]$



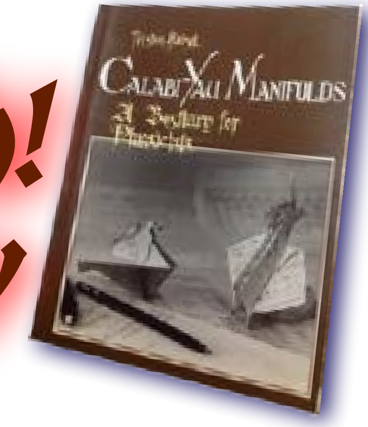




Pre-Historic Prelude
(Where are We Coming From?)

Pre-Historic Prelude

30th! B-day



Classical Constructions — a Summary

nice "ambient space"

- Complete Intersection: $X = \big(\cap_i \{f_i(x)=0\}\big) \subset A = \prod_i \mathbb{P}^{n_i},\ \mathbb{P}^{n_i}_{\vec{w}}$, toric…
- where $f_i(x) \in \Gamma(\mathscr{L}_i)$; $\mathfrak{X}_i = \{f_i(x)=0\} \subset A$

Tian-Yau: $\{\text{Fano}\}_c \smallsetminus \{\text{CY}\}_c = \{\text{CY}\}_{nc}$

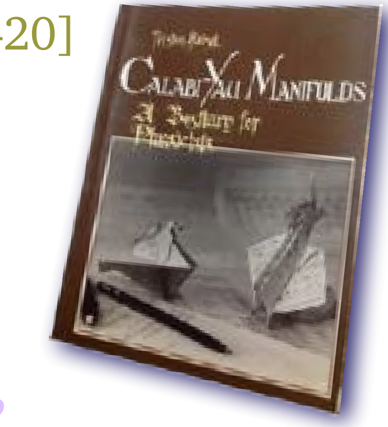
Also: $\{\mathscr{K}^*_{X_c}\} = \{\text{CY}\}_{nc}$

multiplication by

- Koszul resolutions: $\mathscr{L}_i^* \overset{\cdot f_i}{\hookrightarrow} \mathcal{O}_{\cap_{j<i}\mathfrak{X}_j} \twoheadrightarrow \mathcal{O}_{\cap_{j\leqslant i}\mathfrak{X}_j}$
- Adjunction: $T_{\mathfrak{X}_i} \hookrightarrow T_A|_{\mathfrak{X}_i} \overset{\cdot \mathrm{d}f_i}{\twoheadrightarrow} \mathscr{L}_i|_{\mathfrak{X}_i}$ & $T_X \hookrightarrow T_A|_X \overset{\cdot \mathrm{d}(\oplus f_i)}{\twoheadrightarrow} \oplus_i \mathscr{L}_i|_X$
  - Transversality: $\{\wedge_i \mathrm{d}f_i \neq 0\} \cap \{f_i = 0\} \not\subset A$
  - Calabi-Yau: $\det[\oplus_i \mathscr{L}_i] = \mathscr{K}^*_A := \det[T_A] \Leftrightarrow \det[T_X] = \mathcal{O}_X$
  - "Hodge diamond," $H^{p,q}(X) = H^q(X, \wedge^p T^*_X)$, also $H^q(X, \mathrm{End}\,T_X)$
- Long exact cohomology sequences
- Bott-Borel-Weil: $\mathbb{P}^n = \frac{U(n+1)}{U(n)\times U(1)}$, $f_i(x)$ & $H^*(\mathbb{P}^n, \mathscr{L}_i)$ $U(n+1)$-tensors

`+ Macaulay2, SAGE, Magma, ...` (new tricks/old dogs…)

[arXiv:1606.07420]



Pre-Historic Prelude

Classical Constructions

(& smooth$_{\mathbb{R}}$ models) *special? symplectic*

- E.g: $X_m \in \left[\begin{array}{c||c|c} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & m & 2-m \end{array}\right]^{(2,86)}_{-168}$

$b_2 = 2 = h^{1,1}$ dim. space of Kähler classes

$\frac{1}{2}b_3 - 1 = 86 = h^{2,1}$ dim. space of complex structures

$-168 = \chi = 2(h^{1,1} - h^{2,1})$ the Euler #

  - Zero-set of $p(x,y)=0$, $\deg[p] = \binom{1}{m}$, & $q(x,y)=0$, $\deg[q] = \binom{4}{2-m}$
  - Generic $\{p=0\}\cap\{q=0\}$ smooth; $\deg_{\mathbb{P}^n}[p] + \deg_{\mathbb{P}^n}[q] = n+1 \Rightarrow c_1 = 0$
- Sequentially: $X_m \xrightarrow{q=0} \left(F_m \xrightarrow{p=0} \mathbb{P}^4 \times \mathbb{P}^1\right)$ $q(x,y) \overset{?}{\sim} \frac{q_0(x)}{y_0} + \frac{q_1(x)}{y_1}$
  - Chern: $c = \frac{(1+J_1)^5(1+J_2)^2}{(1+J_1+mJ_2)(1+4J_1+(2-m)J_2)} = 1 + [6J_1^{\,2} + (8-3m)J_1J_2] - [20J_1^{\,3} - (32+15mJ_1^{\,2}J_2)]$.
  - C.T.C.Wall: $(aJ_1+bJ_2)^3 = [2a + 3(\underline{4b+ma})]a^2$ $C_{4-k}[(aJ_1+bJ_2)^k] = f_k(\underline{4b+ma})$
  - $p_1[aJ_1+bJ_2] = -88a - 12(\underline{4b+ma})$… the same "$4b+ma$"
  - So, $F_m \approx_{\mathbb{R}} F_{m\,(\mathrm{mod}\,4)}$ & $X_m \approx_{\mathbb{R}} X_{m\,(\mathrm{mod}\,4)}$: 4 *diffeomorphism types*
- …but, $m = 0, 1, 2, 3 \;\Rightarrow\; \deg[q] = \binom{4}{-1}$?!

Meromorphic Madrigal

Why Haven't We Thought of This Before?

[AAGGL:1507.03235 + BH:1606.07420]
[+ GvG:1708.00517]

$$X_m \in \left[\begin{array}{c\|c\|c} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & m & 2-m \end{array}\right]^{(2,86)}_{-168}$$
for $m=3$

- $\deg[q] = \binom{4}{-1}$ holomorphic sections?!
  - Not *everywhere* on $\mathbb{P}^4 \times \mathbb{P}^1$ — (simple poles)
  - but yes on $F_3^{(4)} \subset \mathbb{P}^4 \times \mathbb{P}^1$ — $\geqslant 105$ of 'em!
- How? On $F_3^{(4)}$, $q(x,y) \simeq q(x,y) + \lambda \cdot p(x,y) \leftarrow$ *equivalence class!*
  - [Hirzebruch, 1951]$\Rightarrow$ $p = x_0 y_0{}^3 + x_1 y_1{}^3$ & $q = c(x)\left(\frac{x_0 y_0}{y_1{}^2} - \frac{x_1 y_1}{y_0{}^2}\right)$ $\deg[c] = \binom{3}{0}$
  - So, $q_0 = q(x,y) + \frac{\lambda\, c(x)}{(y_0 y_1)^2} p(x,y) \overset{\lambda \to -1}{=\!=\!=} c(x)\left(-2\frac{x_1 y_1}{y_0{}^2}\right)$ where $y_0 \neq 0$
  - & $q_1 = q(x,y) + \frac{\lambda\, c(x)}{(y_0 y_1)^2} p(x,y) \overset{\lambda \to 1}{=\!=\!=} c(x)\left(2\frac{x_0 y_0}{y_1{}^2}\right)$ where $y_1 \neq 0$
  - & $q_1(x,y) - q_0(x,y) = 2\frac{c(x)}{(y_0 y_1)^2}\, p(x,y) = 0$, on $F_3 := \{p(x,y) = 0\}$
  - [GvG, 1708.00517] scheme-th. "generalized complete intersections"

Reverse-engineered: Mayer-Vietoris sequence & "patching" of the two charts

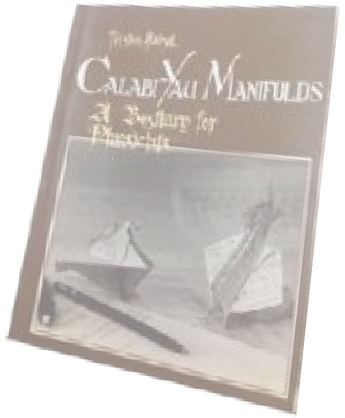
Meromorphic Madrigal

…in well-tempered counterpoint

[BH:1606.07420, 1611.10300 & 2205.12827]
+more

- For $\{\underbrace{x_0 {y_0}^m + x_1 {y_1}^m}_{:=p(x,y;0)} = -\sum_\alpha \epsilon_\alpha \delta p_\alpha(x,y)\} = F^{(n)}_{m;\epsilon} \in \left[\begin{matrix}\mathbb{P}^n \\ \mathbb{P}^1\end{matrix}\middle\|\begin{matrix}1 \\ m\end{matrix}\right]$
  even $p(x,y;0)$ is transverse, $p^{-1}(0)$ is smooth
  - The central ($\epsilon = 0$) member of the family is a Hirzebruch scroll F_m :
  - Directrix: $S := \{\mathfrak{s}(x,y) = 0\}$, $[S] = [H_1] - m[H_2]$ & $[S]^n = -(n-1)m$;
    - where $\mathfrak{s}(x,y) := \big(\frac{x_0}{{y_1}^m} - \frac{x_1}{{y_0}^m}\big) + \frac{\lambda}{(y_0 y_1)^m}[x_0 {y_0}^m + x_1 {y_1}^m]$ degree $\binom{1}{-m}$
  - & $h^0(K^*) = 3\binom{2n-1}{n} + \delta_{\epsilon,0}\vartheta^m_3\binom{2n-2}{2}(m-3)$, $h^0(T) = n^2 + 2 + \delta_{\epsilon,0}\vartheta^m_1(n-1)(m-1)$
  - & $h^1(K^*) = \delta_{\epsilon,0}\vartheta^m_3\binom{2n-2}{2}(m-3)$, $h^1(T) = \delta_{\epsilon,0}\vartheta^m_1(n-1)(m-1)$
  - All these "*exceptionals*" cancel from H^* for $(\epsilon_\alpha \neq 0)$ deformations resulting in *discrete deformations* $F^{(n)}_m \to F^{(n)}_{(m_1,m_2,\cdots)}$ & ⋯ & $\approx_{\mathbb{R}} F^{(n)}_{[m\,(\mathrm{mod}\,n)]}$
- These $F^{(n)}_{(m_1,m_2,\cdots)}$'s are distinct toric varieties… w/$\{\mathfrak{s}_r, r \leqslant m_i\}$

Meromorphic Madrigal



…in well-tempered counterpoint

[BH:1606.07420, 1611.10300 & 2205.12827] +more

- On $F^{(n)}_m$: $p(x,y;0) = x_0 y_0{}^m + x_1 y_1{}^m = 0 \Rightarrow x_0 = -x_1(y_1/y_0)^m$ & $x_1 \to X_1 = \mathfrak{s}$
  - & $(X_i,\ i = 2, \cdots, n+2) = (x_2, \cdots, x_n; y_0, y_1)$
  - $\mathbb{P}^4 \times \mathbb{P}^1$ bi-degree → toric $(\mathbb{C}^\times)^2$-action:
  - BTW, $\det\left[\frac{\partial(p(x,y), \mathfrak{s}(x,y), x_2, \cdots; y_0, y_1)}{\partial(x_0, x_1, x_2, \cdots; y_0, y_1)}\right] = \text{const.}$
  - Need $\deg[f(X)] = \binom{4}{2-m}$, with $\deg[X_1 X_{5,6}^m] = \binom{1}{0} = \deg[X_{2,3,4}]$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow\mathbb{P}^1$

- $f(X) = X_1{}^4 X_{5,6}^{2+3m} \oplus X_1{}^3 X_{2,3,4} X_{5,6}^{2+2m} \cdots \oplus X_1 X_{2,3,4}^3 X_{5,6}^2 \oplus X_{2,3,4}^4 X_{5,6}^{2-m}$
  - $m > 2$,

Meromorphic Madrigal

…in well-tempered counterpoint

[BH:1606.07420, 1611.10300 & 2205.12827]
+more

- On $F_m^{(n)}$: $p(x,y;0) = x_0 y_0{}^m + x_1 y_1{}^m = 0 \Rightarrow x_0 = -x_1(y_1/y_0)^m$ & $x_1 \to X_1 = \mathfrak{s}$
  - & $(X_i,\ i=2,\cdots,n+2) = (x_2,\cdots,x_n;\, y_0, y_1)$
  - $\mathbb{P}^4 \times \mathbb{P}^1$ bi-degree → toric $(\mathbb{C}^\times)^2$-action:
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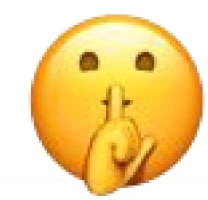


- $f(X) = X_1{}^4 X_{5,6}^{2+3m} \oplus X_1{}^3 X_{2,3,4} X_{5,6}^{2+2m} \cdots \oplus X_1 X_{2,3,4}^3 X_{5,6}^2$

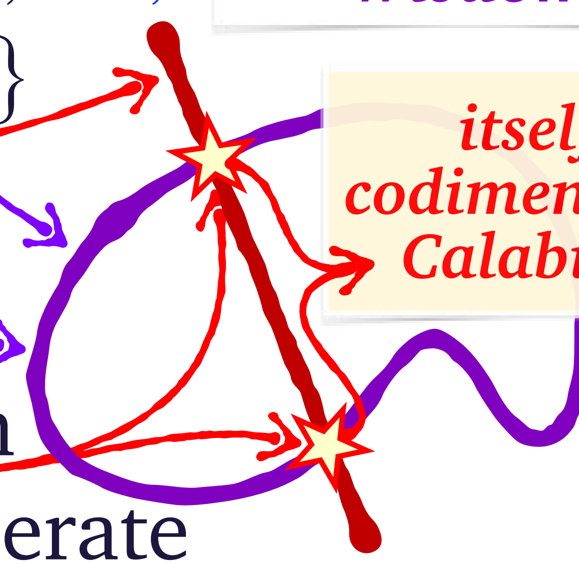
standard wisdom

  - $m > 2$, $\{f(X)=0\} = \{X_1 = 0\} \cup \{\oplus_k X_1{}^k X_{2,3,4}^2 X_{5,6}^{2+km} = 0\}$
  - $\{f(X)=0\}^\sharp = \{X_1 = 0\} \cap \{\oplus_{k=0}^3 X_1{}^k X_{2,3,4}^{4-k} X_{5,6}^{2+km} = 0\}$

$$\left[\begin{array}{c||cc|c} \mathbb{P}^n & 1 & n-1 & 1 \\ \mathbb{P}^1 & m & 2 & -m \end{array}\right] = \left[\begin{array}{c||c|c|c} \mathbb{P}^n & 1 & 1 & n-1 \\ \mathbb{P}^1 & m & -m & 2 \end{array}\right] \overset{\simeq}{\rightarrow} \left[\begin{array}{c||c} \mathbb{P}^{n-2} & n-1 \\ \mathbb{P}^1 & 2 \end{array}\right]$$

$p = 0 = \mathfrak{s} \Leftrightarrow x_0 = 0 = x_1$

Tyurin degenerate



itself a codimension-2 Calabi-Yau

Meromorphic Madrigal

…in well-tempered counterpoint

[BH:1606.07420, 1611.10300 & 2205.12827]
+more

- On $F_m^{(n)}$: $p(x,y;0) = x_0 y_0{}^m + x_1 y_1{}^m = 0 \Rightarrow x_0 = -x_1(y_1/y_0)^m$ & $x_1 \to X_1 = \mathfrak{s}$
  - & $(X_i,\ i=2,\cdots,n+2) = (x_2,\cdots,x_n; y_0, y_1)$
  - $\mathbb{P}^4 \times \mathbb{P}^1$ bi-degree $\to$ toric $(\mathbb{C}^\times)^2$-action:

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
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  - BTW, $\det\left[\frac{\partial(p(x,y), \mathfrak{s}(x,y), x_2, \cdots; y_0, y_1)}{\partial(x_0, x_1, x_2, \cdots; y_0, y_1)}\right] = \text{const.}$
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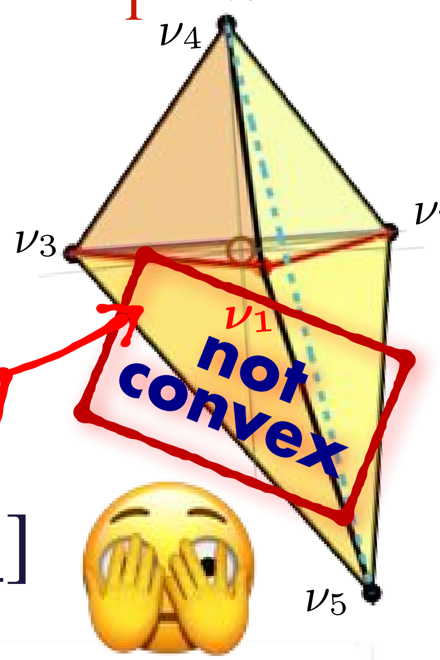
REM*



- $f(X) = X_1{}^4 X_{5,6}^{2+3m} \oplus X_1{}^3 X_{2,3,4} X_{5,6}^{2+2m} \cdots \oplus X_1 X_{2,3,4}^3 X_{5,6}^2 \oplus X_{2,3,4}^4 X_{5,6}^{2-m}$

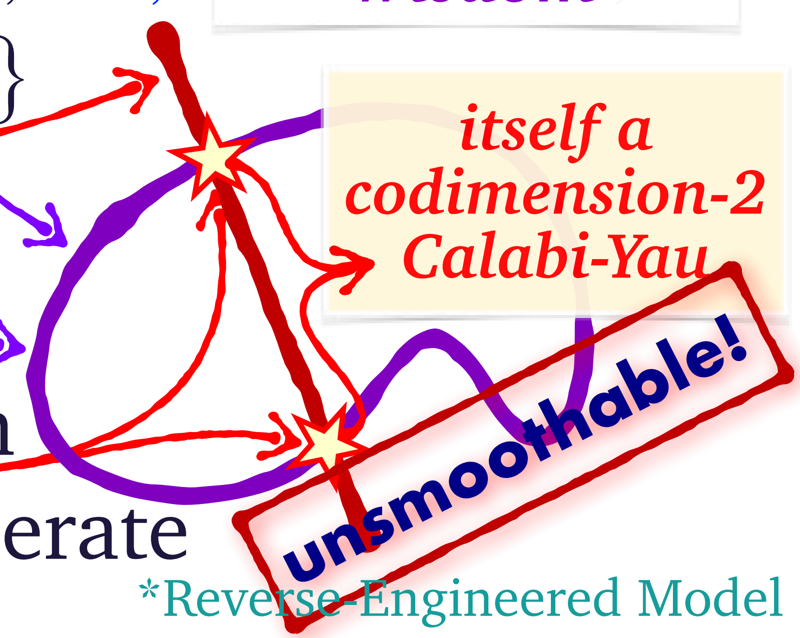
standard wisdom

  - $m > 2$, $\{f(X)=0\} = \{X_1 = 0\} \cup \{\oplus_k X_1{}^k X_{2,3,4}^2 X_{5,6}^{2+km} = 0\}$
  - $\{f(X)=0\}^\sharp = \{X_1=0\} \cap \{\oplus_{k=0}^3 X_1{}^k X_{2,3,4}^{4-k} X_{5,6}^{2+km} = 0\}$

$$\left[\begin{array}{c\|cc\|c} \mathbb{P}^n & 1 & n-1 & 1 \\ \mathbb{P}^1 & m & 2 & -m \end{array}\right] = \left[\begin{array}{c\|c\|c\|c} \mathbb{P}^n & 1 & 1 & n-1 \\ \mathbb{P}^1 & m & -m & 2 \end{array}\right] \xrightarrow{\simeq} \left[\begin{array}{c\|c} \mathbb{P}^{n-2} & n-1 \\ \mathbb{P}^1 & 2 \end{array}\right]$$

$p = 0 = \mathfrak{s} \Leftrightarrow x_0 = 0 = x_1$

Tyurin degenerate



*Reverse-Engineered Model

Meromorphic Minuet

…with a meandering melody

[BH:1606.07420, 1611.10300 & 2205.12827]
+more

- Algorithm:

Construction 2.1 *Given a degree-$\binom{1}{m}$ hypersurface $\{p_{\vec\epsilon}(x,y)\,0\} \subset \mathbb{P}^n \times \mathbb{P}^1$ as in (2.2), construct*

$$\deg = \binom{1}{m-r_0-r_1}: \quad \mathfrak{s}_{\vec\epsilon}(x,y;\lambda) := \mathop{\rm Flip}_{y_0}\Big[\frac{1}{{y_0}^{r_0}\,{y_1}^{r_1}}\,p_{\vec\epsilon}(x,y)\Big] \pmod{p_{\vec\epsilon}(x,y)}, \qquad \left[\begin{matrix}\mathbb{P}^n\\ \mathbb{P}^1\end{matrix}\middle\|\begin{matrix}1\\ m\end{matrix}\right]$$

progressively decreasing $r_0{+}r_1 = 2m, 2m{-}1, \cdots$, and keeping only Laurent polynomials containing both y_0- and y_1-denominators but no y_0, y_1-mixed ones. The "Flip_{y_i}" operator changes the relative sign of the rational monomials with y_i-denominators. For algebraically independent such sections, restrict to a subset with maximally negative degrees that are not overall (y_0, y_1)-multiples of each other.

- E.g.: $m=2$

```
p0 = x0 y0^2 + x1 y1^2; ep[α_] := Table[1/(y0^(α-i) y1^i), {i, 0, α}]; Expand /@ (p0 {ep[5], ep[4], ep[3]})
```

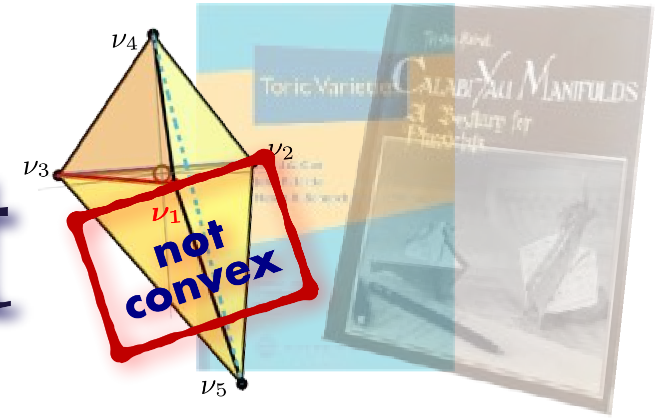
$$\Big\{\Big\{\frac{x_0}{y_0^3}+\frac{x_1 y_1^2}{y_0^5},\ \frac{x_0}{y_0^2 y_1}+\frac{x_1 y_1}{y_0^4},\ \frac{x_1}{y_0^3}+\frac{x_0}{y_0 y_1^2},\ \frac{x_0}{y_1^3}+\frac{x_1}{y_0^2 y_1},\ \frac{x_0 y_0}{y_1^4}+\frac{x_1}{y_0 y_1^2},\ \frac{x_0 y_0^2}{y_1^5}+\frac{x_1}{y_1^3}\Big\},$$
$$\Big\{\frac{x_0}{y_0^2}+\frac{x_1 y_1^2}{y_0^4},\ \frac{x_0}{y_0 y_1}+\frac{x_1 y_1}{y_0^3},\ \frac{x_1}{y_0^2}+\frac{x_0}{y_1^2},\ \frac{x_0 y_0}{y_1^3}+\frac{x_1}{y_0 y_1},\ \frac{x_0 y_0^2}{y_1^4}+\frac{x_1}{y_1^2}\Big\},\ \Big\{\frac{x_0}{y_0}+\frac{x_1 y_1^2}{y_0^3},\ \frac{x_0}{y_1}+\frac{x_1 y_1}{y_0^2},\ \frac{x_1}{y_0}+\frac{x_0 y_0}{y_1^2},\ \frac{x_0 y_0^2}{y_1^3}+\frac{x_1}{y_1}\Big\}\Big\}$$

$\cdot\, y_1, \quad \cdot\, y_0$

- finds $\mathfrak{s}(x,y) = \Big(\frac{x_0}{{y_1}^2} - \frac{x_1}{{y_0}^2}\Big) \bmod (x_0{y_0}^2 + x_1{y_1}^2)$; $\deg = \binom{1}{-2}$, $[\mathfrak{s}^{-1}(0)] = [J_1] - 2[J_2]$.

THE **exceptional curve $[S]^2 = -1$ in $F_2^{(2)}$**

Meromorphic Minuet

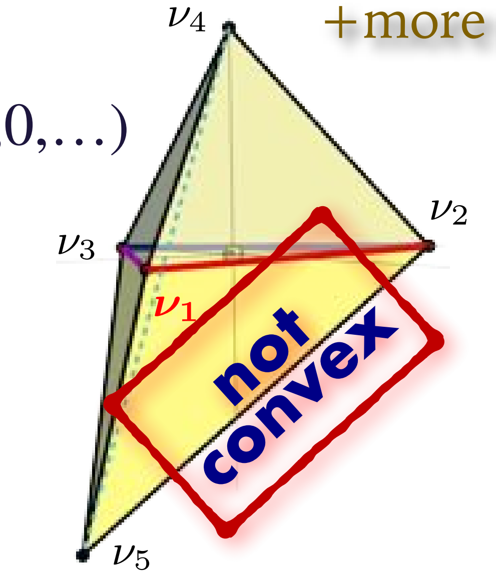


…with a meandering melody

[BH:1606.07420, 1611.10300 & 2205.12827]
+more

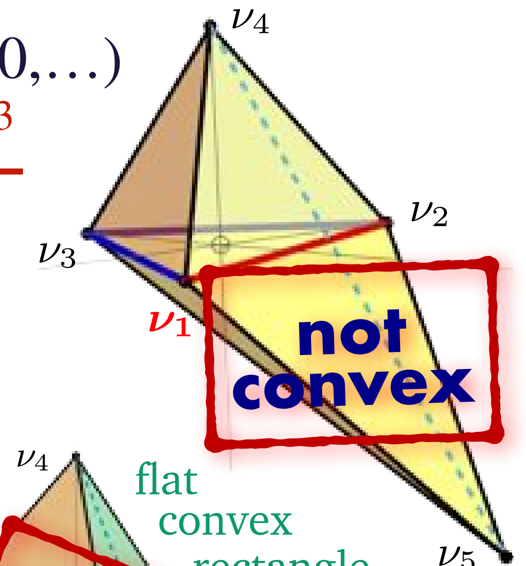
- Deform: $p_1(x,y) = x_0 y_0{}^5 + x_1 y_1{}^5 + x_2 y_0 y_1{}^4$ toric $F^{(n)}_{(4,1,0,\ldots)}$
  - Find: $\mathfrak{s}_{1,1}(x,y) = \frac{x_0 y_0}{y_1{}^5} + \frac{x_2}{y_1{}^4} - \frac{x_1}{y_1{}^4}$ & $\mathfrak{s}_{1,2}(x,y) = \frac{x_0}{y_1} - \frac{x_2}{y_0} - \frac{x_1 y_1{}^4}{y_0{}^5}$
  - & $\det\left[\frac{\partial(p_1, \mathfrak{s}_{1,1}, \mathfrak{s}_{1,2}, x_3, \cdots; y_0, y_1)}{\partial(x_0, x_1, x_2, x_3, \cdots; y_0, y_1)}\right] = \text{const.}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow \mathbb{P}^4$
−4	−1	0	0	1	1	$\leftarrow \mathbb{P}^1$



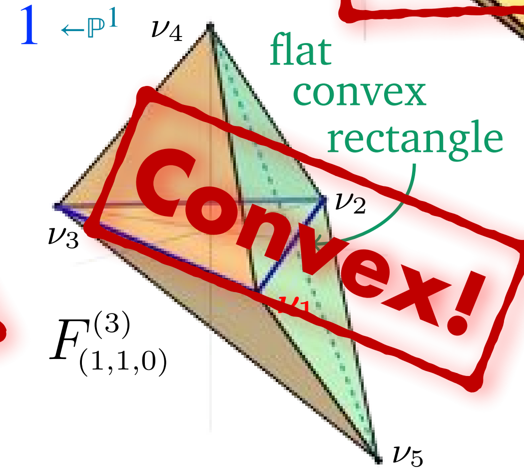
- Deform: $p_2(x,y) = x_0 y_0{}^5 + x_1 y_1{}^5 + x_2 y_0{}^2 y_1{}^3$ toric $F^{(n)}_{(3,2,0,\ldots)}$
  - Find: $\mathfrak{s}_{2,1}(x,y) = \frac{x_0 y_0{}^2}{y_1{}^5} + \frac{x_2}{y_1{}^3} - \frac{x_1}{y_1{}^3}$ & $\mathfrak{s}_{2,2}(x,y) = \frac{x_0}{y_1{}^2} - \frac{x_2}{y_0{}^2} - \frac{x_1 y_1{}^3}{y_0{}^5}$
  - & $\det\left[\frac{\partial(p_2, \mathfrak{s}_{2,1}, \mathfrak{s}_{2,2}, x_3, \cdots; y_0, y_1)}{\partial(x_0, x_1, x_2, x_3, \cdots; y_0, y_1)}\right] = \text{const.}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow \mathbb{P}^4$
−3	−2	0	0	1	1	$\leftarrow \mathbb{P}^1$



- … and $p_3(x,y) = x_0 y_0{}^5 + x_1 y_1{}^5 + x_2 y_0{}^2 y_1{}^3 + x_3 y_0{}^3 y_1{}^2$
  - → toric $F^{(n)}_{(2,2,1,\cdots)}$ for $n=3$, $F^{(3)}_{(2,2,1)} \approx F^{(3)}_{(1,1,0)}$

Fano!



Meromorphic March

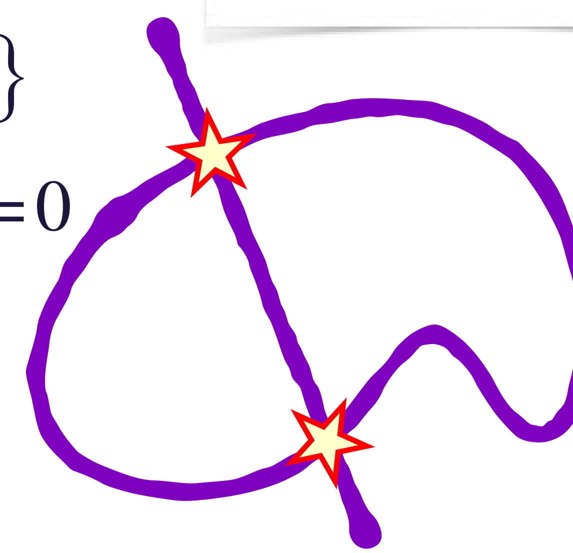
…back to the medial motif

- On $F_m^{(n)}$: $x_0 y_0{}^m + x_1 y_1{}^m = 0 \;\Rightarrow\; x_0 = -x_1 (y_1/y_0)^m$ & $x_1 \to X_1 = \mathfrak{s}$
  - & $(X_i,\ i=2,\cdots,n+2) = (x_2,\cdots,x_n; y_0, y_1)$
  - $\mathbb{P}^4 \times \mathbb{P}^1$ bi-degree $\to$ toric $(\mathbb{C}^\times)^2$-action:
  - BTW, $\det\left[\dfrac{\partial(p(x,y),\mathfrak{s}(x,y),x_2,\cdots;y_0,y_1)}{\partial(x_0,x_1,x_2,\cdots;y_0,y_1)}\right] = \text{const.}$
  - Need $[f(X)] = \binom{4}{2-m}$, with $\deg[X_1 X_{5,6}^m] = \binom{1}{0} = \deg[X_{2,3,4}]$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow \mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow \mathbb{P}^1$

- $f(X) = X_1{}^4 X_{5,6}^{2+3m} \oplus X_1{}^3 X_{2,3,4} X_{5,6}^{2+2m} \cdots \oplus X_1 X_{2,3,4}^3 X_{5,6}^2 \oplus X_{2,3,4}^4 X_{5,6}^{2-m}$
  - $m > 2$, $\{f(X)=0\} = \{X_1 = 0\} \cup \{\oplus_k X_1{}^k X_{2,3,4}^2 X_{5,6}^{2+km} = 0\}$
  - $\{f(X)=0\}^\sharp = \{X_1 = 0\} \cap \{\oplus_k X_1{}^k X_{2,3,4}^2 X_{5,6}^{2+km} = 0\}$: $R_{\mu\nu} = 0$

standard wisdom



Meromorphic March

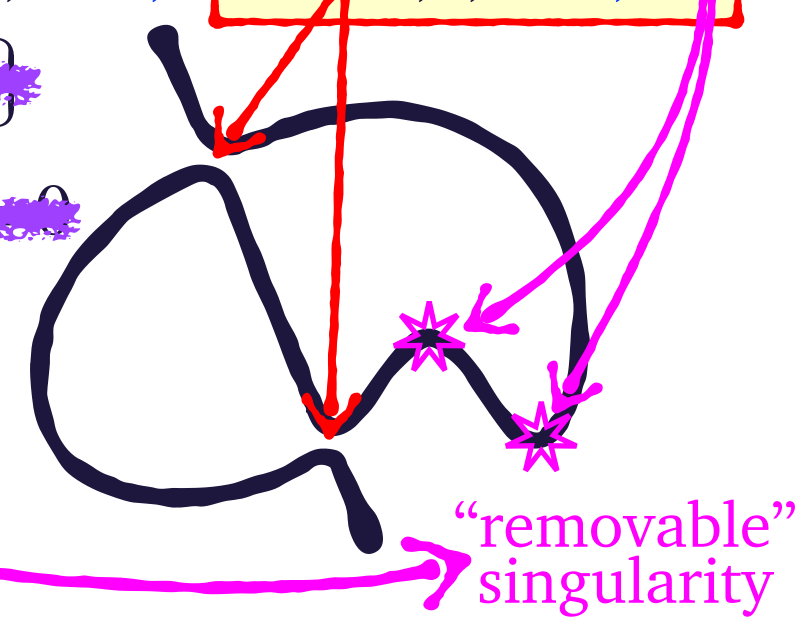
1611.10300 & 2205.12827 +much more

…back to the medial motif

- On $F_m^{(n)}$: $x_0 y_0{}^m + x_1 y_1{}^m = 0 \;\Rightarrow\; x_0 = -x_1 (y_1/y_0)^m$ & $x_1 \to X_1 = \mathfrak{s}$
  - & $(X_i,\ i=2,\cdots,n+2) = (x_2,\cdots,x_n; y_0, y_1)$
  - $\mathbb{P}^4 \times \mathbb{P}^1$ bi-degree → toric $(\mathbb{C}^\times)^2$-action:
  - BTW, $\det\left[\frac{\partial(p(x,y),\mathfrak{s}(x,y),x_2,\cdots;y_0,y_1)}{\partial(x_0,x_1,x_2,\cdots;y_0,y_1)}\right] = \text{const.}$
  - Need $[f(X)] = \binom{4}{2-m}$, with $\deg[X_1 X_{5,6}^m] = \binom{1}{0} = \deg[X_{2,3,4}]$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow\mathbb{P}^1$

- $f(X) = X_1{}^4 X_{5,6}^{2+3m} \oplus X_1{}^3 X_{2,3,4} X_{5,6}^{2+2m} \cdots \oplus X_1 X_{2,3,4}^3 X_{5,6}^2 \oplus X_{2,3,4}^4 X_{5,6}^{2-m}$
  - $m>2$, $\{f(X)=0\}$ ~~$\{X_1=0\} \cup \{\oplus_k X_1{}^k X_{2,3,4}^2 X_{5,6}^{2+km}=0\}$~~
  - ~~$\{f(X)=0\}^\sharp = \{X_1=0\} \cap \{\oplus_k X_1{}^k X_{2,3,4}^2 X_{5,6}^{2+km}=0\} = B_{\mu\nu}$~~
  - Embrace the Laurent terms = transverse
  - "Intrinsic limit" (L'Hôpital-"repaired") → smooth *(pre?)* complex spaces



Meromorphic March

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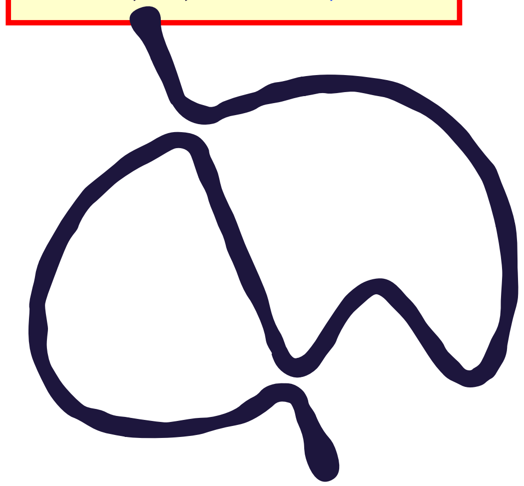
…back to the medial motif

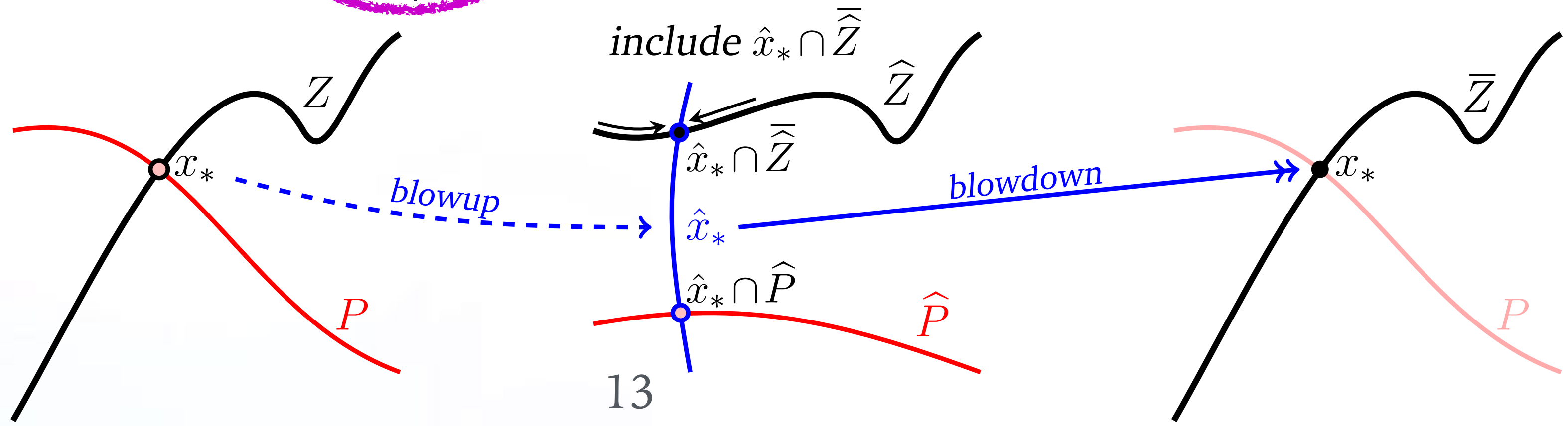
- $f(X) = X_1{}^4 X_{5,6}^{2+3m} \oplus X_1{}^3 X_{2,3,4} X_{5,6}^{2+2m} \cdots \oplus X_1 X_{2,3,4}^3 X_{5,6}^2 \oplus \boxed{X_{2,3,4}^4 X_{5,6}^{2-m}}$
  - $m > 2$, Laurent terms & "intrinsic limit" ⁉️



- "Intrinsic limit" (L'Hopital's rule)
  - Toy example: $f(x) = x_3{}^5 + x_4{}^5 + \frac{x_2{}^2}{x_4} = 0$ near $x_4 = 0$
  - Well, *away* from $x_4 = 0$, $x_3{}^5 + x_4{}^5 + \frac{x_2{}^2}{x_4} = 0$ is well and spry
  - so $x_2{}^2 = -(x_3{}^5 x_4 + x_4{}^6)_{x_4 \neq 0} \;\mapsto\; x_2 \overset{f(x)=0}{=\!=\!=} x_2(x_3, x_4)$
  - Then, $\lim_{\substack{x_4 \to 0 \\ f(x)=0}} \left(x_3{}^5 + x_4{}^5 + \frac{x_2(x_3, x_4)^2}{x_4}\right) = (x_3{}^5) + (0) + (-x_3{}^5) = 0$

just like $\lim_{x \to 0} \frac{\sin(x)}{x}$

- Or, maybe:

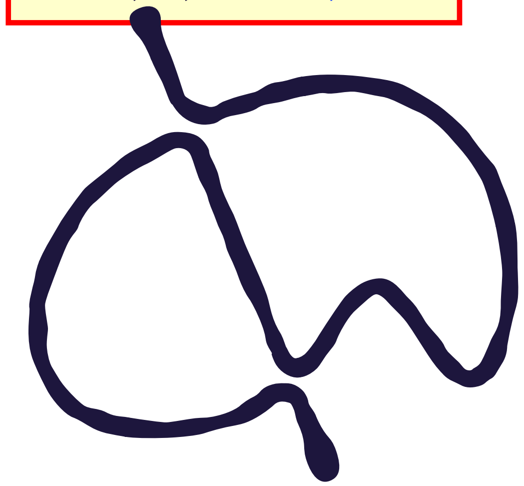


Meromorphic March

1611.10300 & 2205.12827
+much more

…back to the medial motif

- $f(X) = X_1^{\,4} X_{5,6}^{2+3m} \oplus X_1^{\,3} X_{2,3,4} X_{5,6}^{2+2m} \cdots \oplus X_1 X_{2,3,4}^3 X_{5,6}^2 \oplus X_{2,3,4}^4 X_{5,6}^{2-m}$
  - $m > 2$, Laurent terms & "intrinsic limit" !?



- Virtual varieties [F. Severi] [A. Gholampour], i.e., Weil divisors
  - E.g., $\mathbb{P}^2_{(3:1:1)}[5]$: $0 = x_3{}^5 + x_4{}^5 + \frac{x_2{}^2}{x_4} = \frac{x_3{}^5 x_4 + x_4{}^6 + x_2{}^2}{x_4}$
  - Denominator contributions tend to subtract from those of the numerator
- Change variables [David Cox] [H. Schenck]: $(x_2, x_3, x_4) \mapsto (z_3\sqrt{z_2}, z_1{}^2, z_2)$
  - $x_3{}^5 + x_4{}^5 + \frac{x_2{}^2}{x_4} \mapsto z_1{}^{10} + z_2{}^5 + z_3{}^2$ in $\mathbb{P}^2_{(1:2:5)}[10]$
  - **Generalized to all** $F_m^{(n)}[c_1]$ ✅ — *not a fluke*
  - A *desingularized* *finite quotient* of a *branched multiple cover*
  - …and a variety of "general type" ($c_1 < 0$ or even $c_1 \gtrless 0$)

…there's ∞ of those, just as of VEX polytopes!

Meromorphic March

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+much more

…back to the medial motif

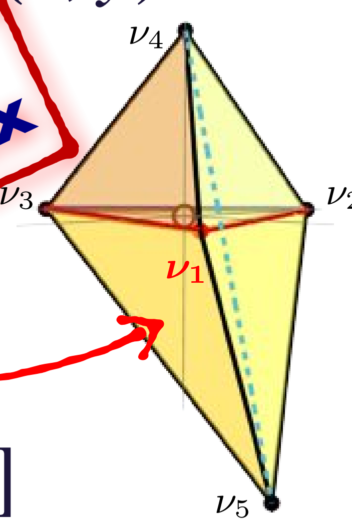
- On $F_m^{(n)}$: $x_0 {y_0}^m + x_1 {y_1}^m = 0$; $\det\left[\frac{\partial(p(x,y), \mathfrak{s}(x,y), x_2, \cdots; y_0, y_1)}{\partial(x_0, x_1, x_2, \cdots; y_0, y_1)}\right]$ = const. & $p(x,y) = 0$.
  - $\mathbb{P}^n \times \mathbb{P}^1$-degrees → Mori vectors

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow \mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow \mathbb{P}^1$

  - central in family $F_{m;\epsilon}^{(n)} \in \left[\begin{matrix}\mathbb{P}^n \\ \mathbb{P}^1\end{matrix}\middle\| \begin{matrix}1 \\ m\end{matrix}\right]$
  - deformations $p(x,y;\epsilon) := p(x,y;0) + \sum_\alpha \epsilon_\alpha \delta p_\alpha$
  - have less non-convex sp. polytopes & less singular $\Gamma[\mathscr{K}^*(F_{\vec{m}}^{(n)})]$

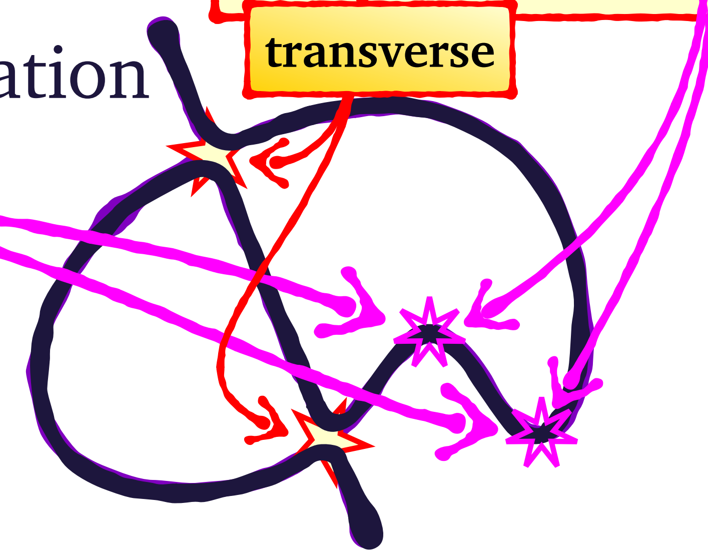
not convex

REM*



- $f(X) = {X_1}^4 X_{5,6}^{2+3m} \oplus {X_1}^3 X_{2,3,4} X_{5,6}^{2+2m} \cdots \oplus X_1 X_{2,3,4}^3 X_{5,6}^2 \oplus X_{2,3,4}^4 X_{5,6}^{2-m}$
  - $m > 2$, regular ↦ "unsmoothable" Turin degeneration
  - Laurent smoothing (w/L'Hôpital repair)
  - CY = Weyl divisors in non-Fano
  - desingularized finite quotient of branched multiple covers ↔ general type var's

transverse



Laurent-Toric Fugue

(a *not-so-new* Toric Geometry)

A Generalized Construction of
Calabi-Yau Mirror Models

arXiv:1611.10300 + 2205.12827

+ lots more…

Laurent-Toric Fugue

& Non-Convex Mirrors $m=3$ —2D *Proof-of-Concept*—

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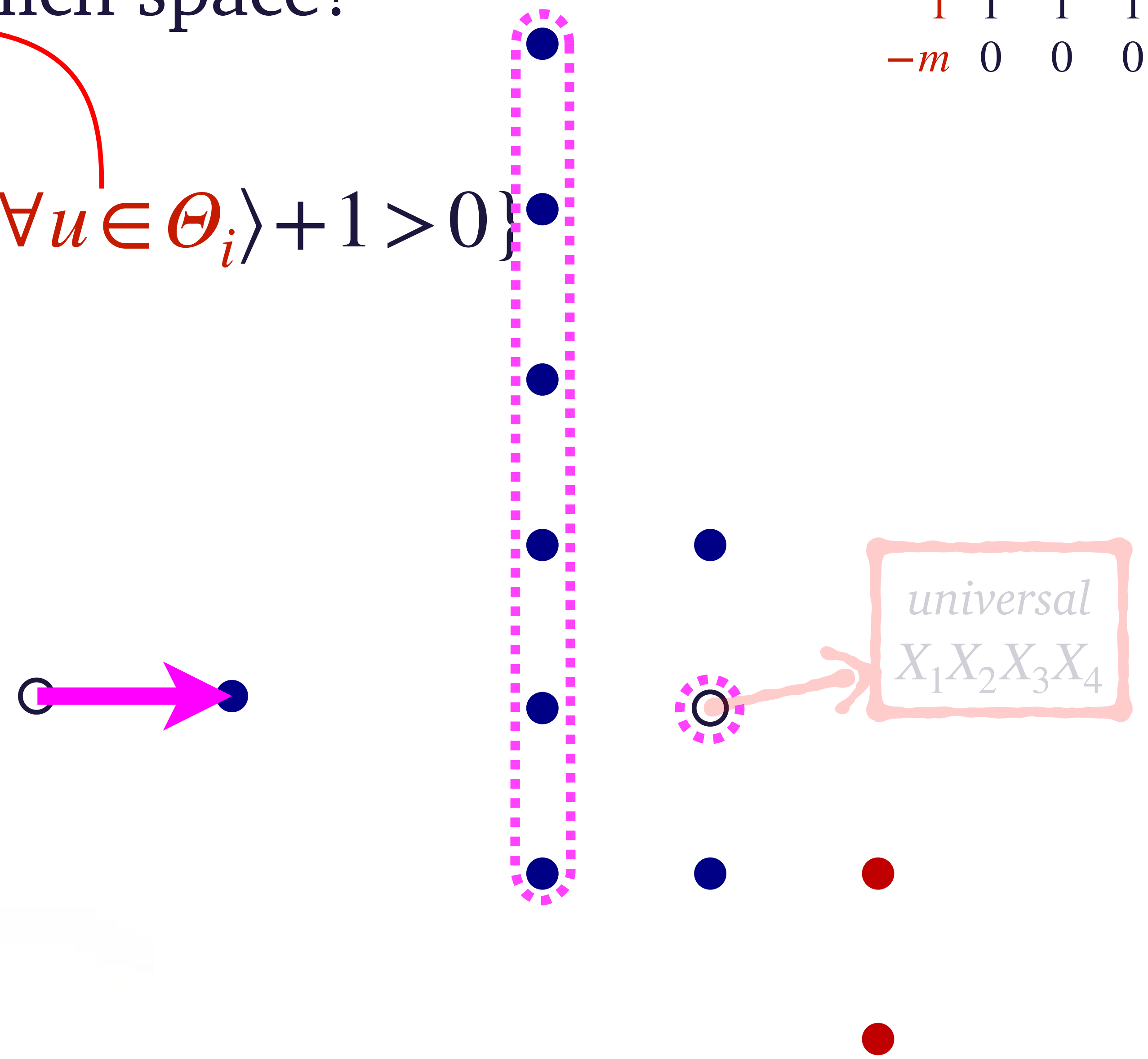
+much more

- $X_1{}^2 X_2{}^0 (X_3 \oplus X_4)^{2+1m} \oplus X_1{}^1 X_2{}^1 (X_3 \oplus X_4)^{2+0m} \oplus X_1{}^0 X_2{}^2 (X_3 \oplus X_4)^{2-1m}$
- *Trans*polar: functions on which space?
  - $\Delta \to \bigcup_i (\text{convex } \Theta_i)$;
  - Compute $\Theta_i \to \Theta_i^\circ := \{v\!: \langle v | \forall u \in \Theta_i \rangle + 1 > 0\}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow \mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow \mathbb{P}^1$



Laurent-Toric Fugue

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& Non-Convex Mirrors $m=3$ —2D *Proof-of-Concept*—

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X_1	X_2	X_3	X_4	X_5	X_6	
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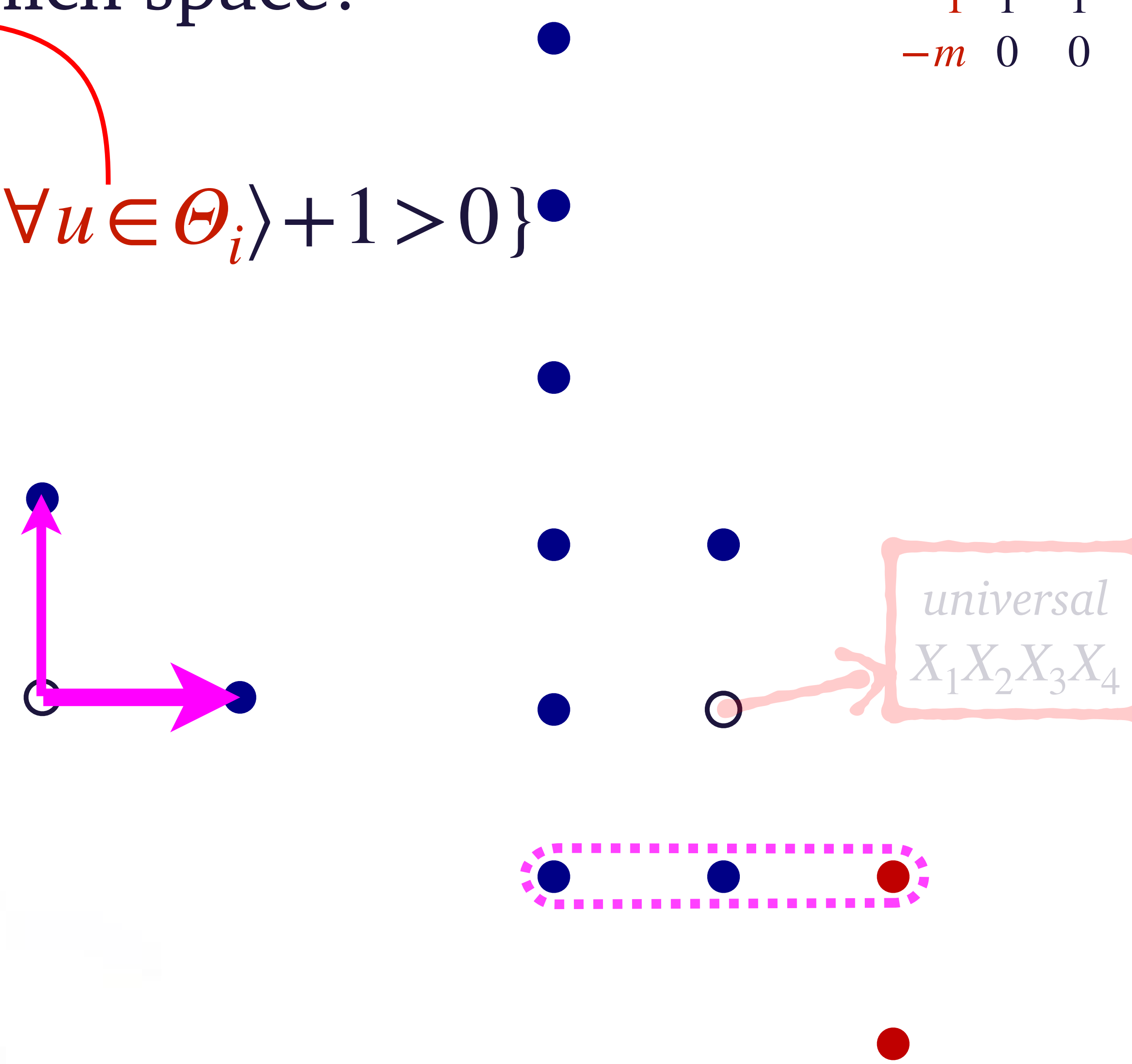

Laurent-Toric Fugue

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+much more

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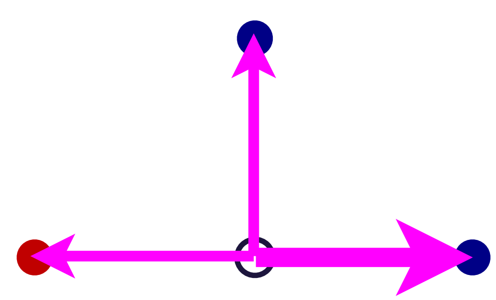
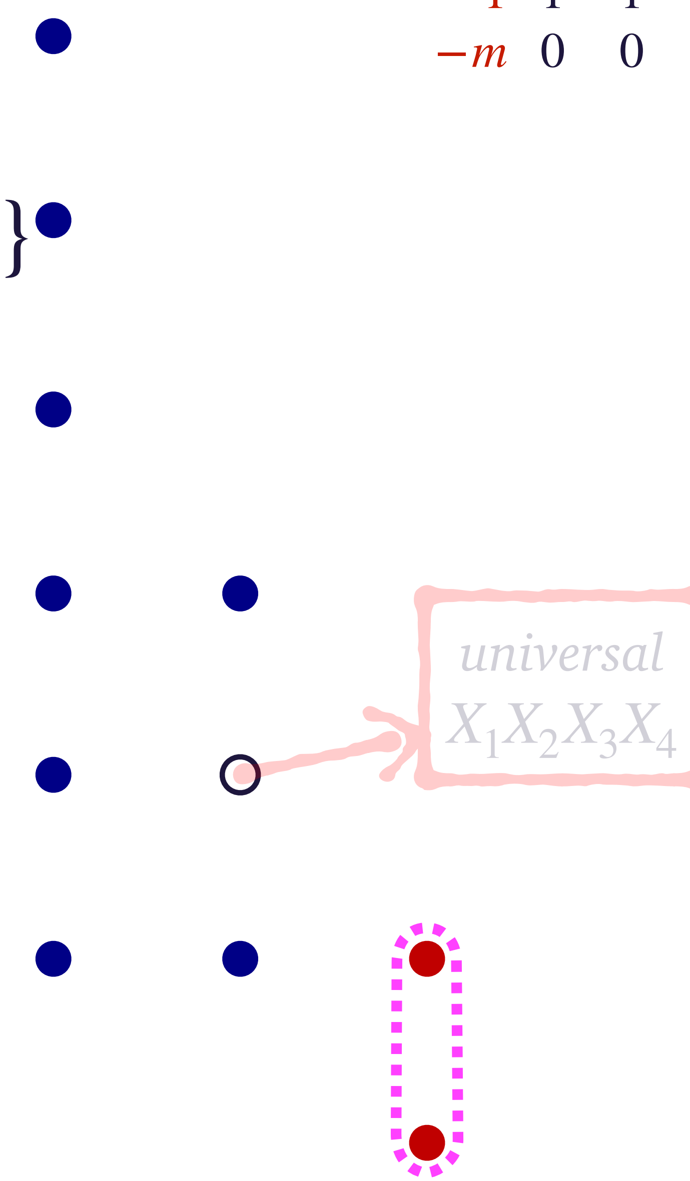
X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow \mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow \mathbb{P}^1$





Laurent-Toric Fugue

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+much more

& Non-Convex Mirrors $m=3$ —2D *Proof-of-Concept*—

- $X_1{}^2 X_2{}^0 (X_3 \oplus X_4)^{2+1m} \oplus X_1{}^1 X_2{}^1 (X_3 \oplus X_4)^{2+0m} \oplus X_1{}^0 X_2{}^2 (X_3 \oplus X_4)^{2-1m}$

X_1	X_2	X_3	X_4	X_5	X_6	
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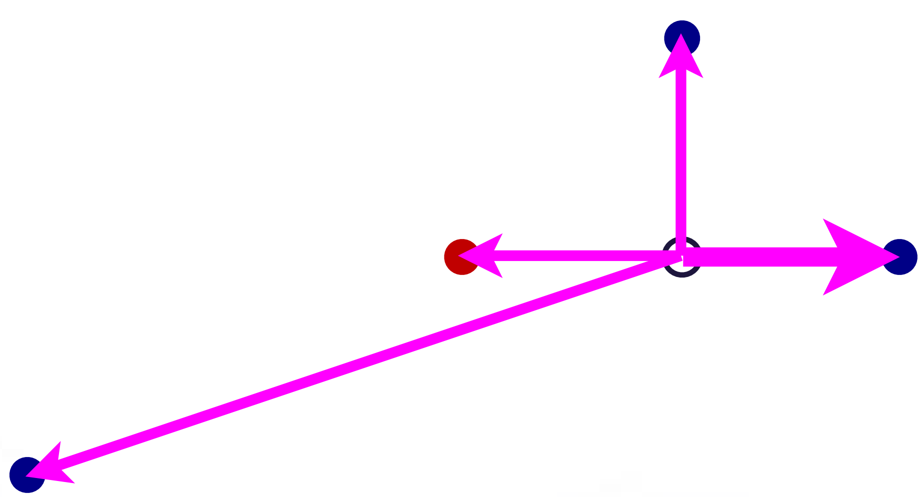

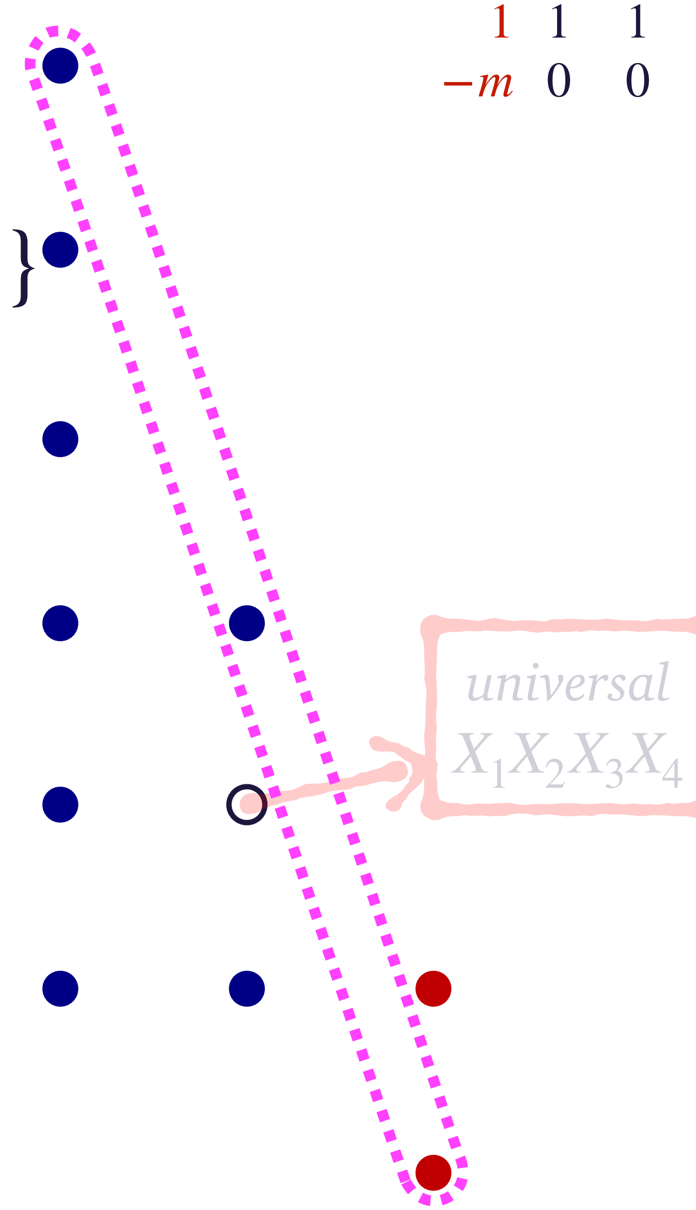

Laurent-Toric Fugue

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+much more

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X_1	X_2	X_3	X_4	X_5	X_6	
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  - $\Delta \to \bigcup_i (\text{convex } \Theta_i)$;
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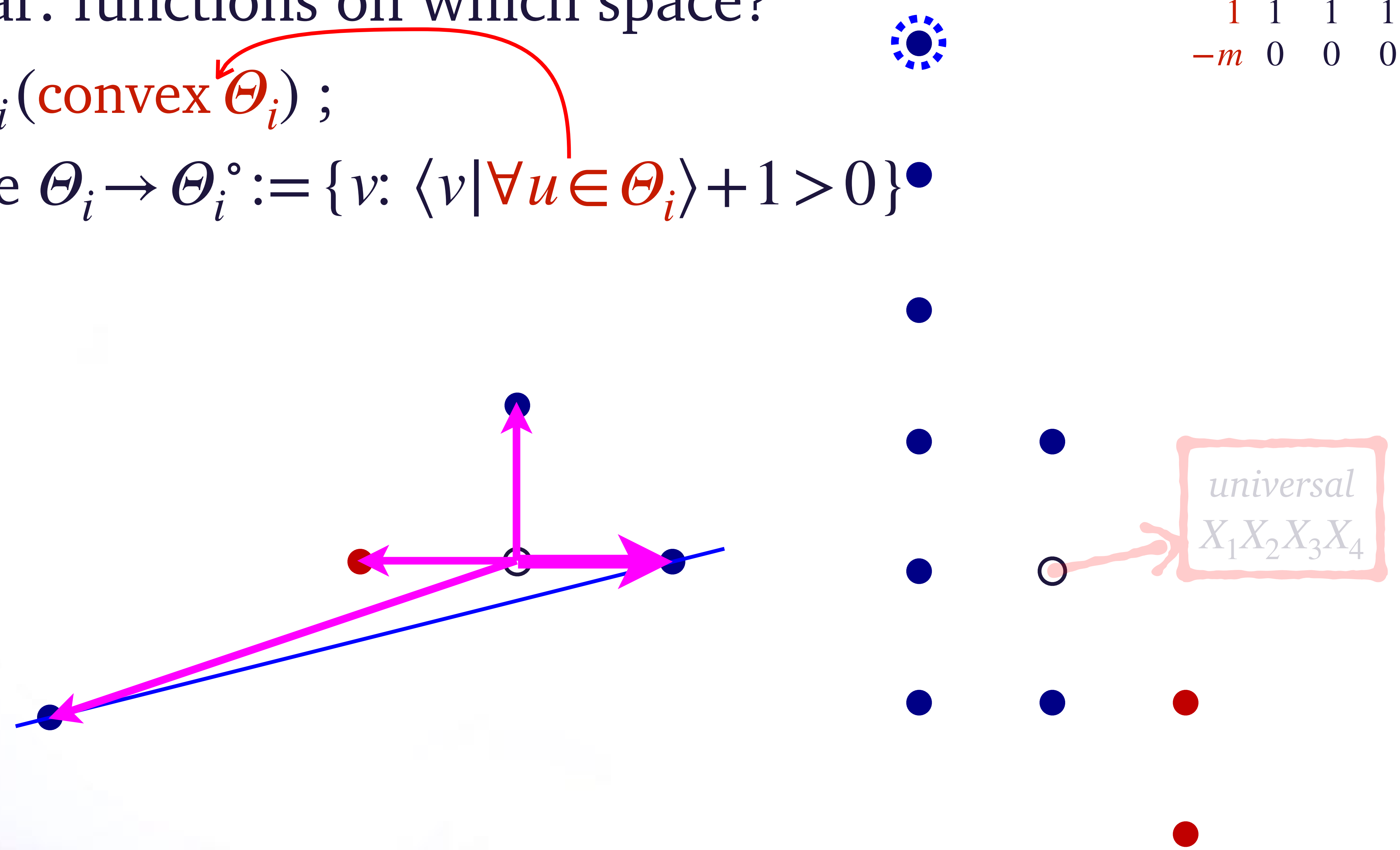

Laurent-Toric Fugue

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& Non-Convex Mirrors

$m=3$ —2D *Proof-of-Concept*—

- $X_1^{\,2}X_2^{\,0}(X_3\oplus X_4)^{2+1m}\oplus X_1^{\,1}X_2^{\,1}(X_3\oplus X_4)^{2+0m}\oplus X_1^{\,0}X_2^{\,2}(X_3\oplus X_4)^{2-1m}$
- *Trans*polar: functions on which space?
  - $\Delta \to \bigcup_i(\text{convex } \Theta_i)$;
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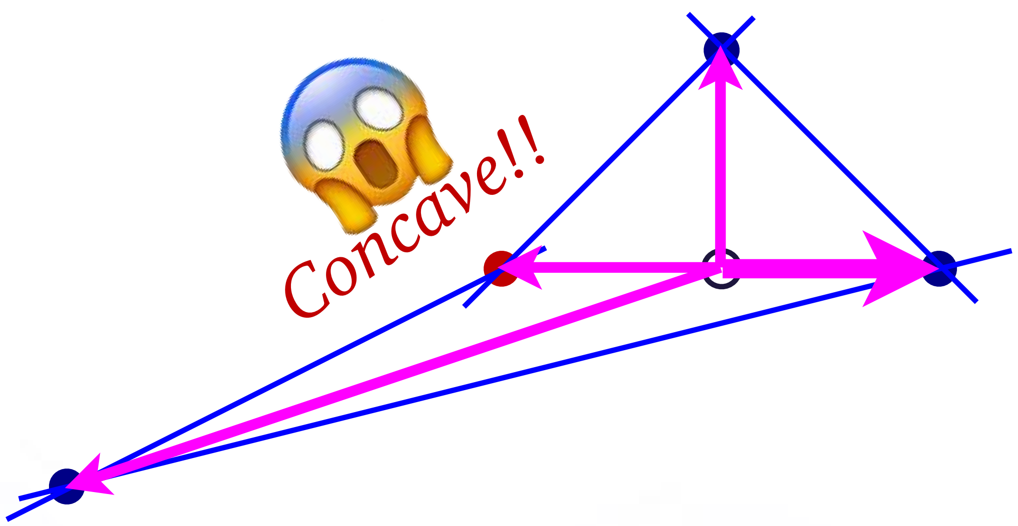
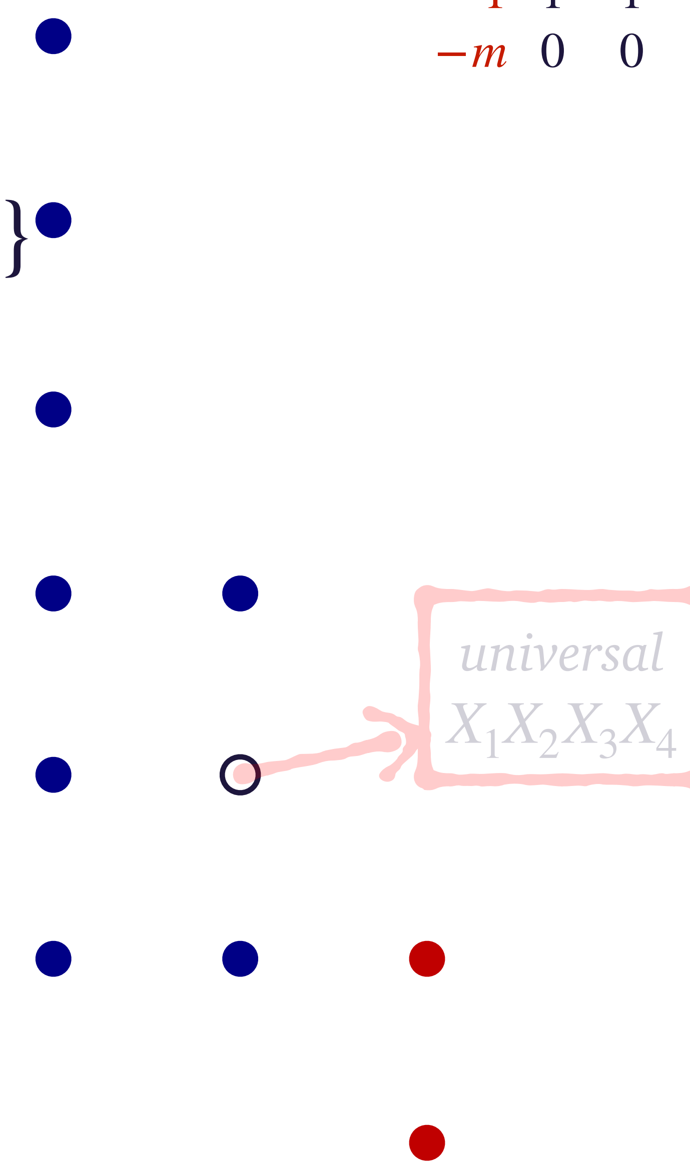
X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow\mathbb{P}^1$





Laurent-Toric Fugue

& Non-Convex Mirrors

—2D *Proof-of-Concept*—

$m=3$

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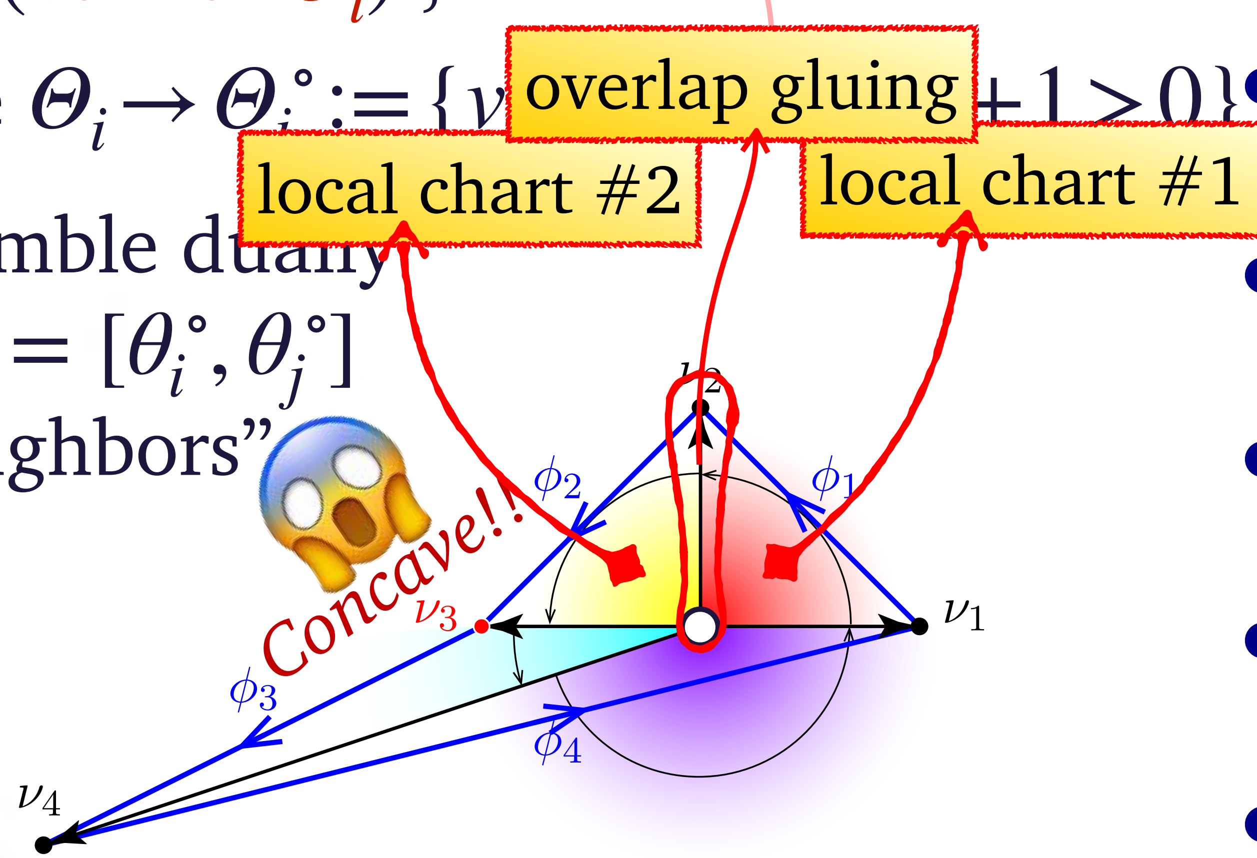
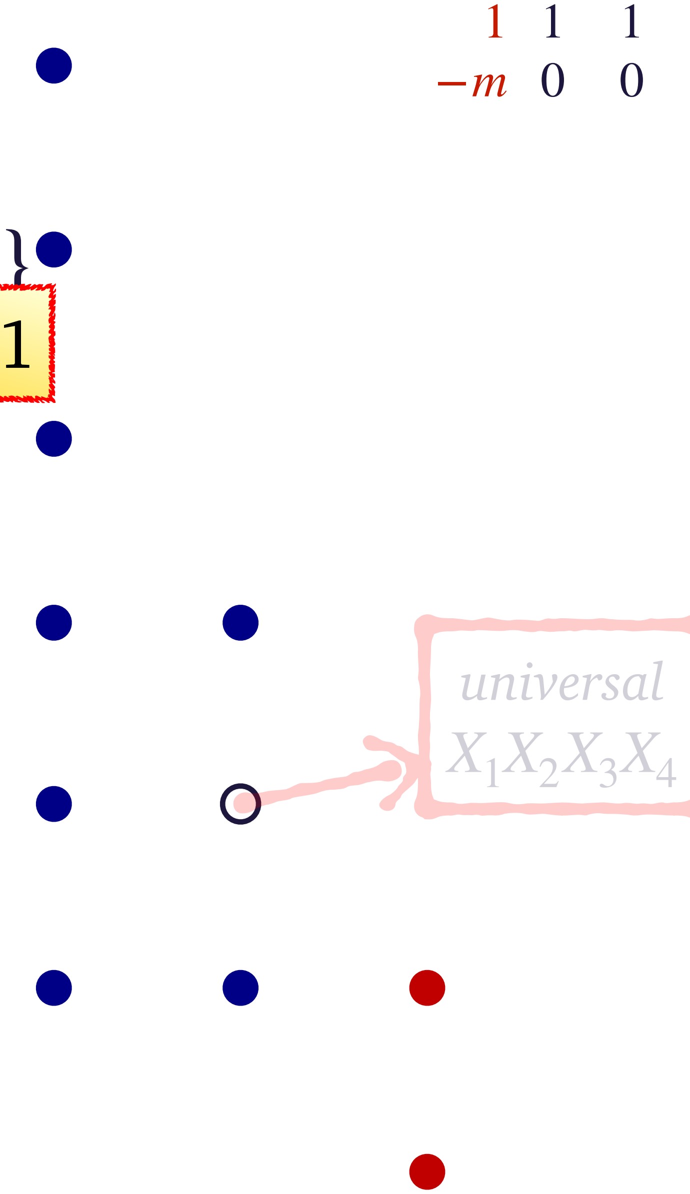
X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow\mathbb{P}^1$

- *Trans*polar: functions on which space?
  - $\Delta\to\bigcup_i(\text{convex }\Theta_i)$;
  - Compute $\Theta_i\to\Theta_i^\circ:=\{\nu$ [illegible] $+1>0\}$
  - (Re)assemble du[illegible]
    $(\theta_i\cap\theta_j)^\circ=[\theta_i^\circ,\theta_j^\circ]$
    with "neighbors"





Laurent-Toric Fugue

& Non-Convex Mirrors $m=3$ —2D *Proof-of-Concept*—

- $X_1^{\,2}X_2^{\,0}(X_3\oplus X_4)^{2+1m}\oplus X_1^{\,1}X_2^{\,1}(X_3\oplus X_4)^{2+0m}\oplus X_1^{\,0}X_2^{\,2}(X_3\oplus X_4)^{2-1m}$

X_1	X_2	X_3	X_4	X_5	X_6	
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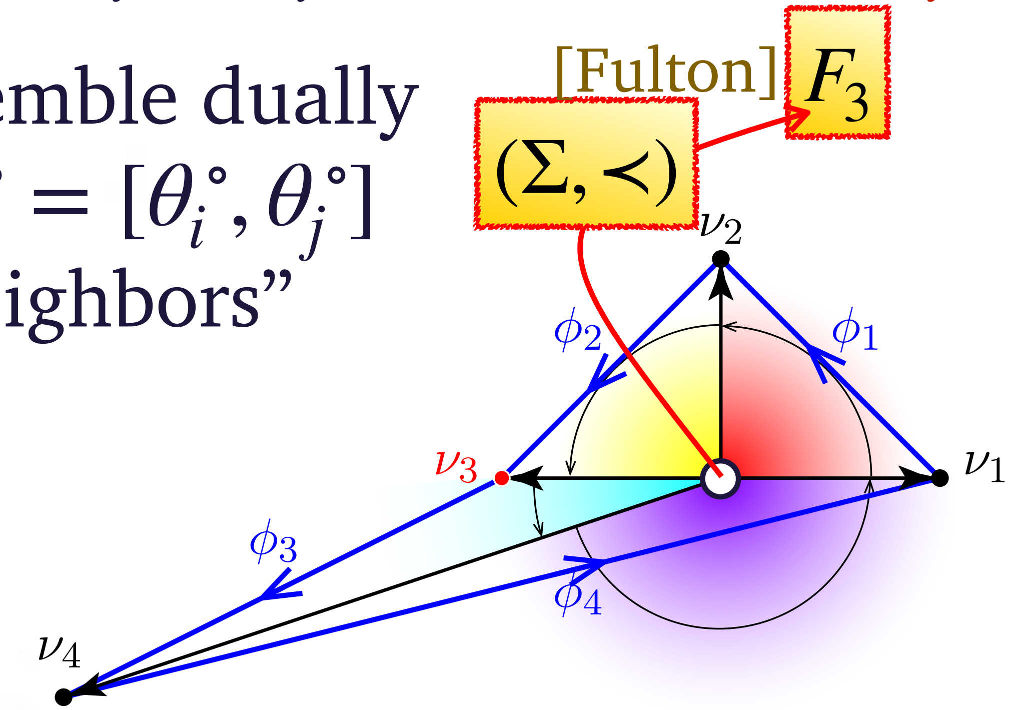
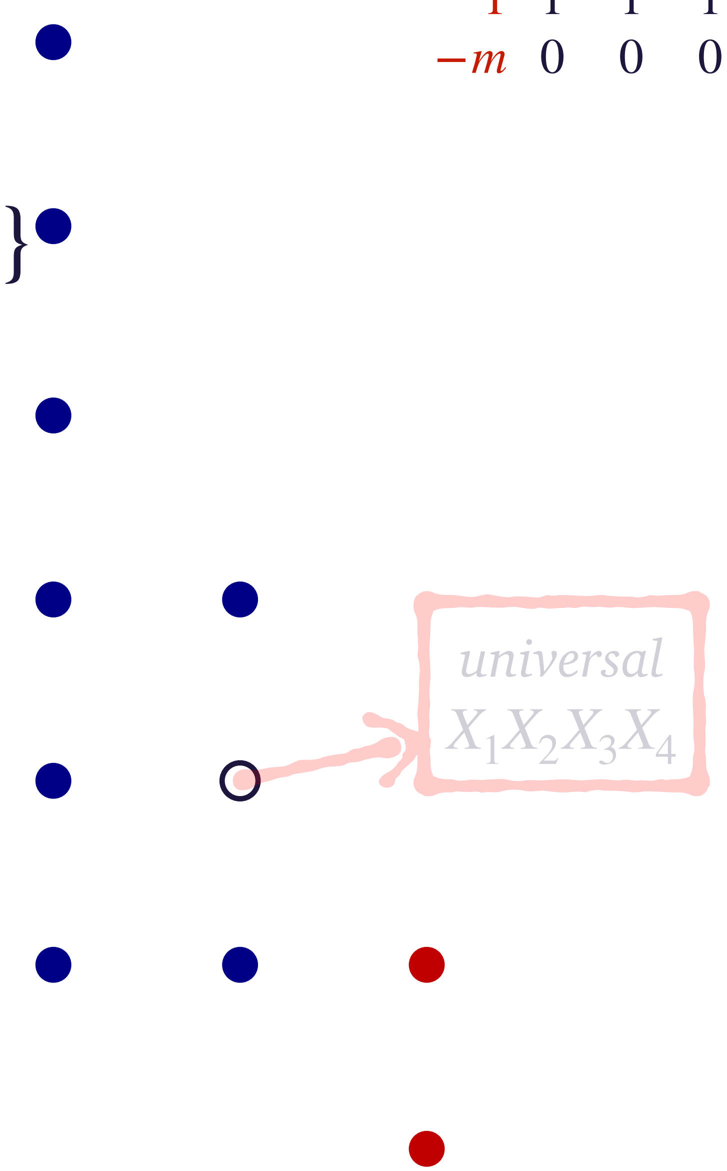
  - (Re)assemble dually $(\theta_i\cap\theta_j)^\circ=[\theta_i^\circ,\theta_j^\circ]$ with "neighbors"





Laurent-Toric Fugue

& Non-Convex Mirrors

—2D *Proof-of-Concept*—

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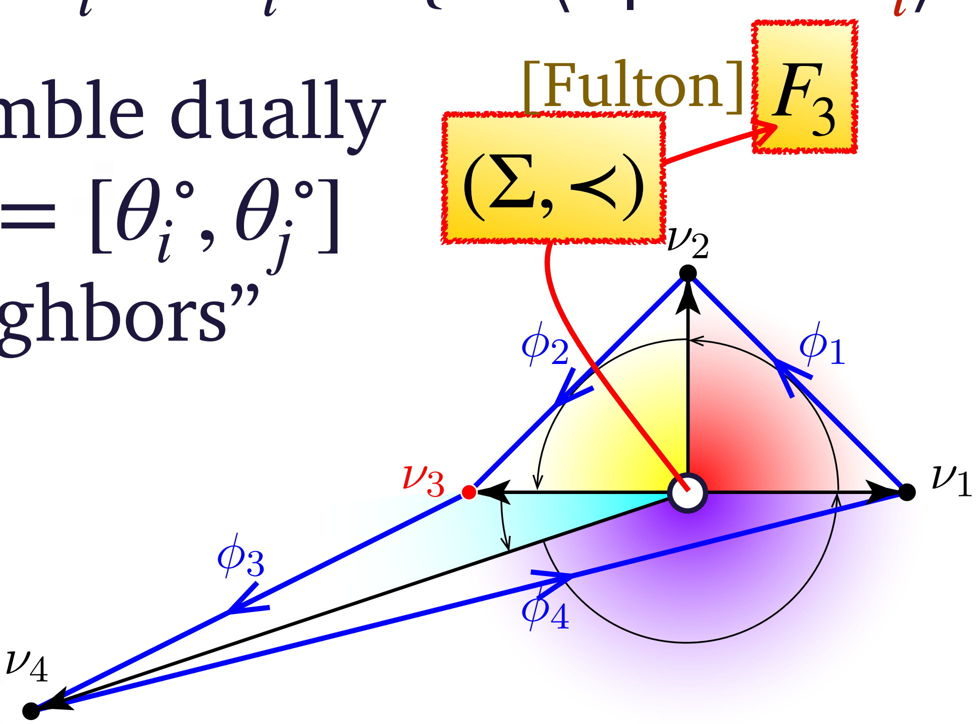
+much more

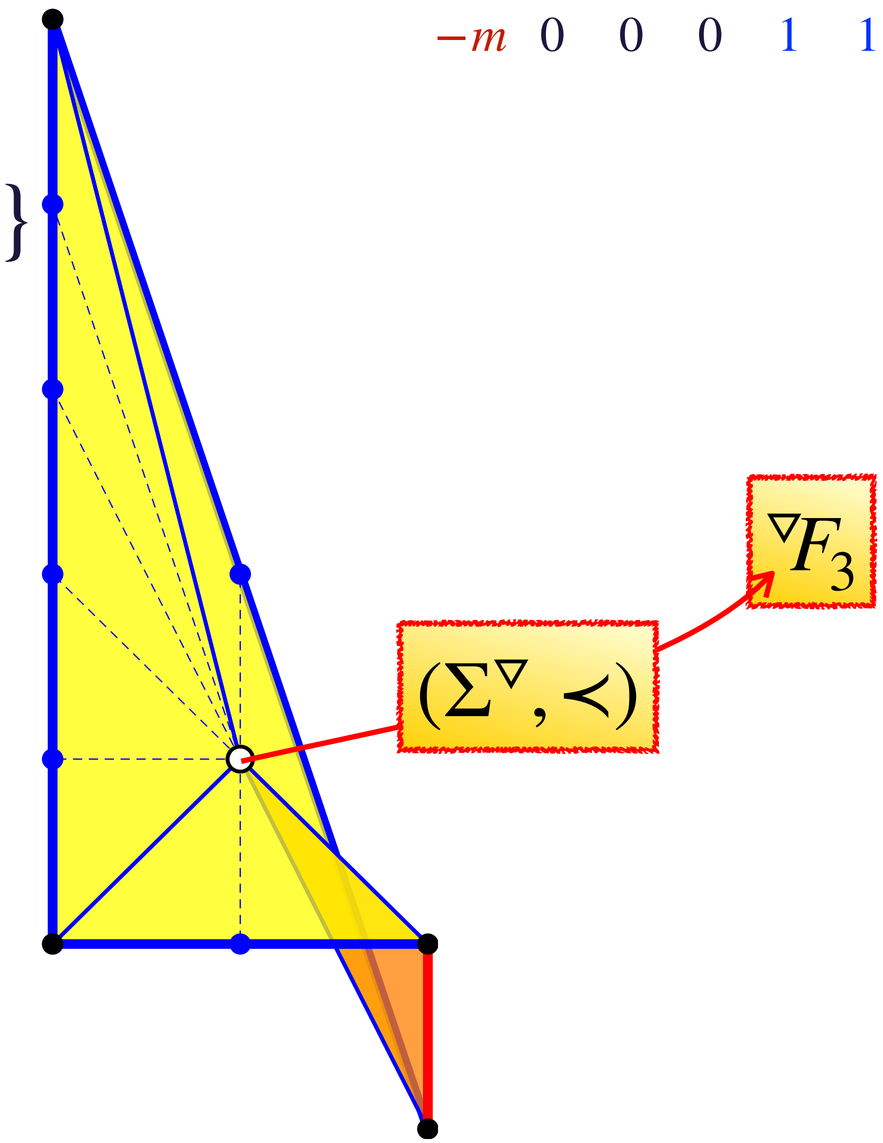
$m=3$

- $X_1^{\,2}X_2^{\,0}(X_3\oplus X_4)^{2+1m}\oplus X_1^{\,1}X_2^{\,1}(X_3\oplus X_4)^{2+0m}\oplus X_1^{\,0}X_2^{\,2}(X_3\oplus X_4)^{2-1m}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow\mathbb{P}^1$

- *Trans*polar: functions on which space?
  - $\Delta\to\bigcup_i(\text{convex }\Theta_i)$;
  - Compute $\Theta_i\to\Theta_i^\circ:=\{v\!:\langle v|\forall u\in\Theta_i\rangle+1>0\}$
  - (Re)assemble dually $(\theta_i\cap\theta_j)^\circ=[\theta_i^\circ,\theta_j^\circ]$ with "neighbors"





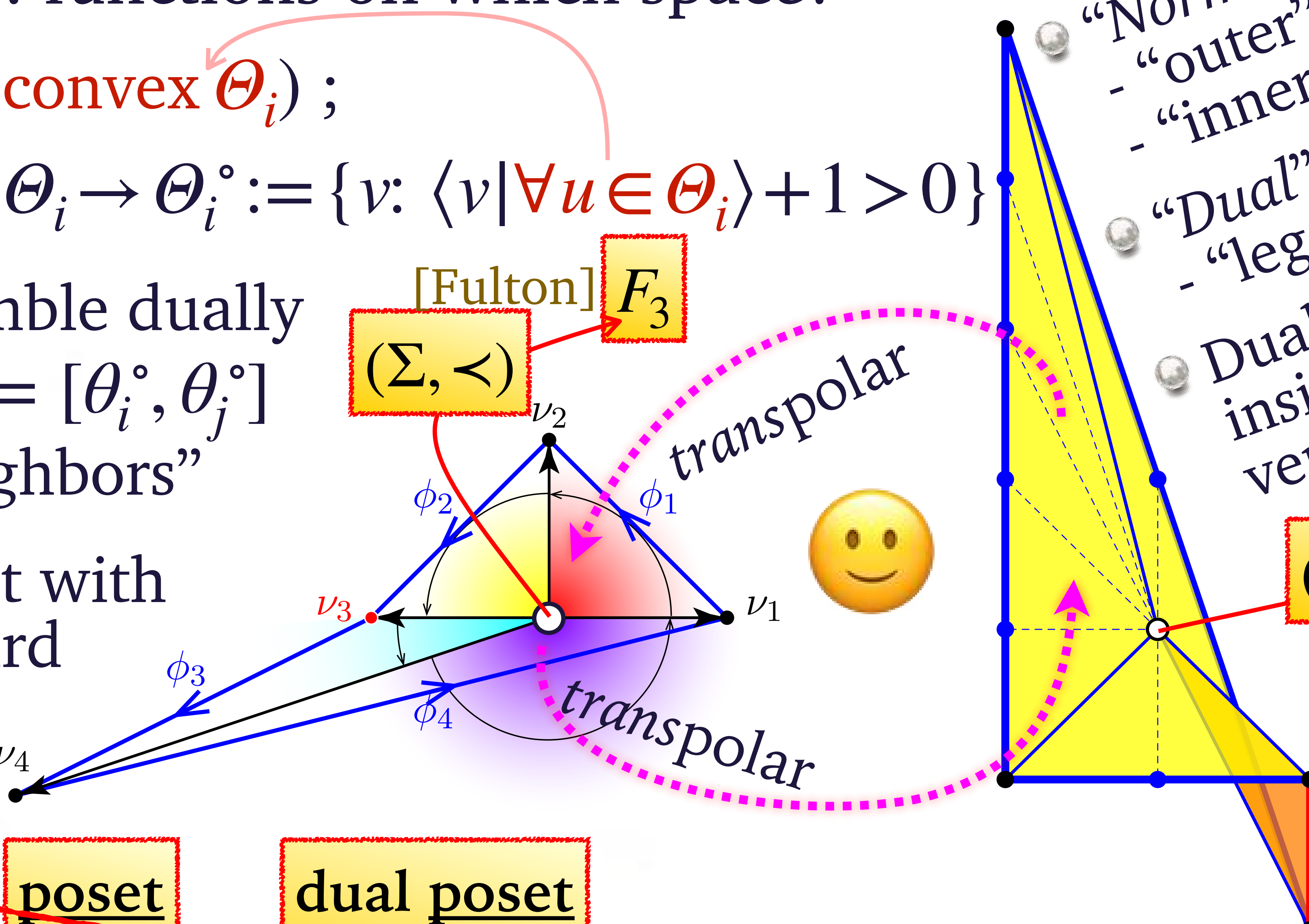
Laurent-Toric Fugue

1611.10300 & 2205.12827
+much more

& Non-Convex Mirrors $m=3$ —2D *Proof-of-Concept*—

- $X_1^{\,2}X_2^{\,0}(X_3\oplus X_4)^{2+1m}\oplus X_1^{\,1}X_2^{\,1}(X_3\oplus X_4)^{2+0m}\oplus X_1^{\,0}X_2^{\,2}(X_3\oplus X_4)^{2-1m}$
- *Trans*polar: functions on which space?
  - $\Delta\to\bigcup_i(\text{convex }\Theta_i)$;
  - Compute $\Theta_i\to\Theta_i^{\circ}:=\{v\!: \langle v|\forall u\in\Theta_i\rangle+1>0\}$
  - (Re)assemble dually $(\theta_i\cap\theta_j)^{\circ}=[\theta_i^{\circ},\theta_j^{\circ}]$ with "neighbors"
  - Consistent with all standard methods

- "Normal fan"
  - "outer" [GE]
  - "inner/local" [C,L&S]
- "Dual"
  - "legal loops" [P&RV]
- Dual cones ↦ inside opening vertex-cones [?BH]



(pre)complex algebraic geometry

poset $(\Sigma,\prec)$ ⟷ **dual poset** $(\Sigma^{\nabla},\prec)$

$F_3[c_1]\overset{\text{MM}}{\longleftrightarrow}{}^{\nabla}\!F_3[c_1]$ → (pre)symplectic geometry

'92: Khovanskii +Pukhlikov
'93: Karshon +Tolman
'99: Hattori +Masuda
+lots of

Laurent-Toric Fugue

& Non-Convex Mirrors

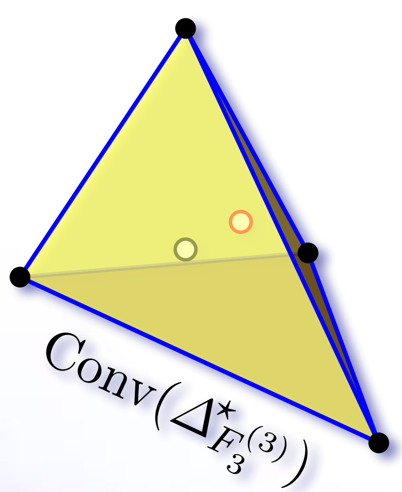
—3D *Proof-of-Concept*—

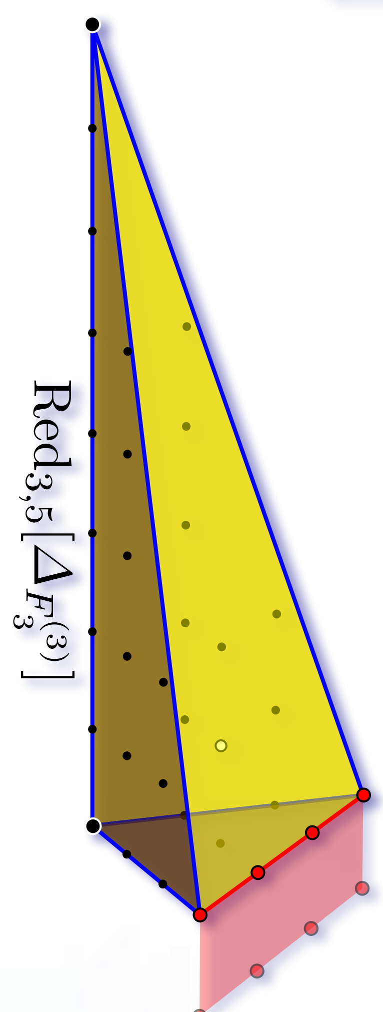
- (Toric) transposition:

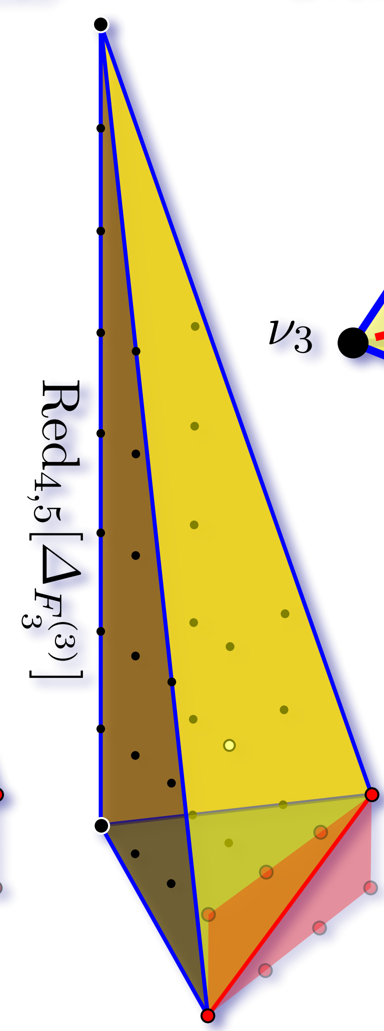
$$f(x;\Delta_{F_m^{(3)}}) = a_1\,x_1{}^3\,x_4{}^{2m+2} + a_2\,x_1{}^3\,x_5{}^{2m+2} + a_3\,\frac{x_2{}^3}{x_4{}^{m-2}} + a_4\,\frac{x_2{}^3}{x_5{}^{m-2}} + a_5\,\frac{x_3{}^3}{x_4{}^{m-2}} + a_6\,\frac{x_3{}^3}{x_5{}^{m-2}}$$

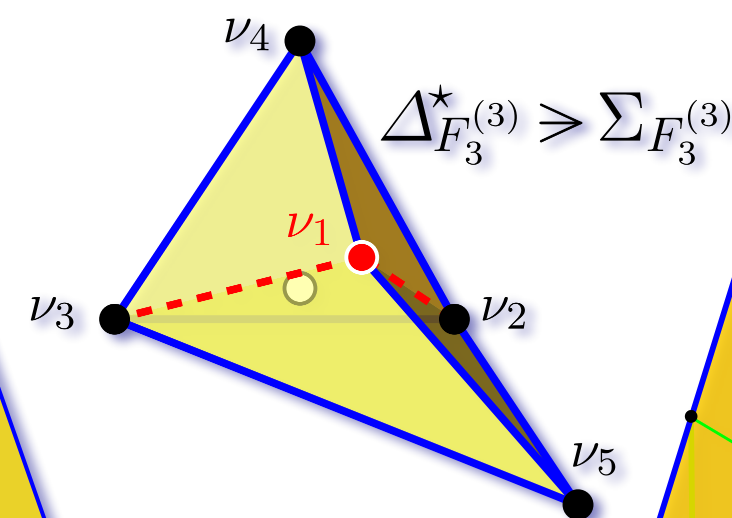
$$g(y;\Delta^\star_{F_m^{(3)}}) = \underbrace{b_1\,y_1{}^3\,y_2{}^3}_{\nu_1} + b_2\,y_3{}^3\,y_4{}^3 + b_3\,y_5{}^3\,y_6{}^3 + b_4\,\frac{y_1{}^{2m+2}}{(y_3\,y_5)^{m-2}} + b_5\,\frac{y_2{}^{2m+2}}{(y_4\,y_6)^{m-2}}$$

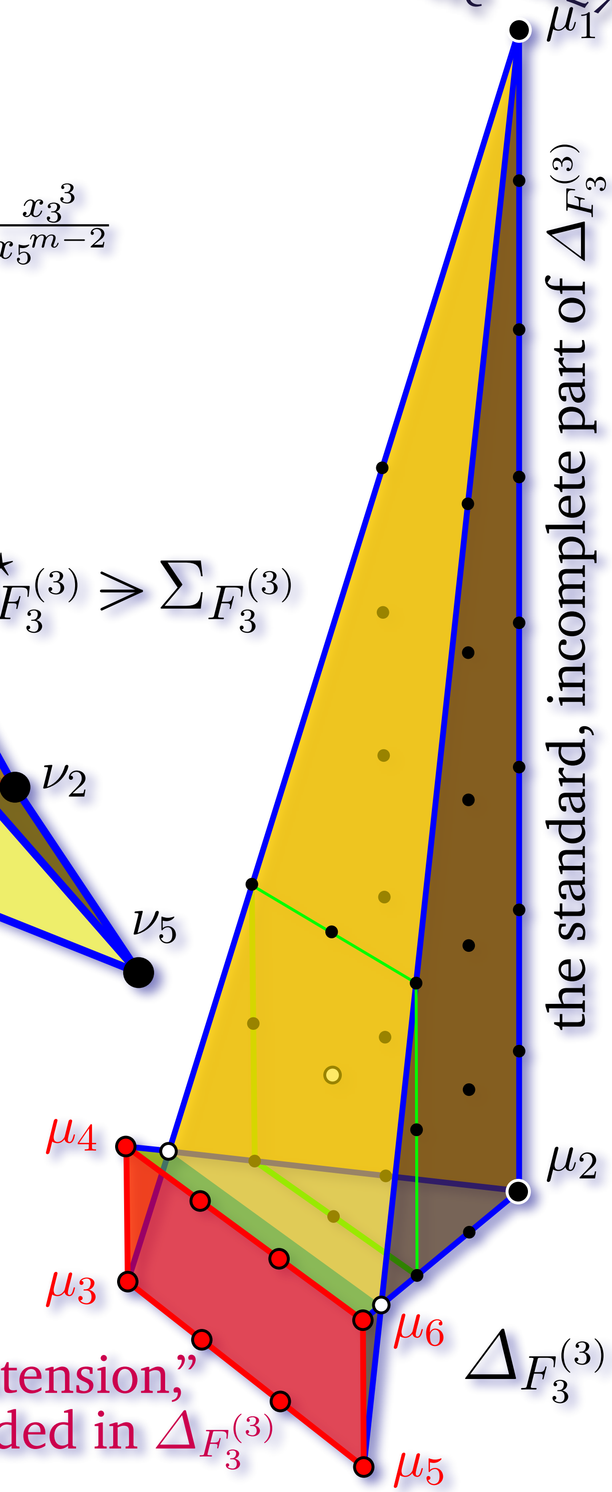
$$\mathbb{E} = \begin{bmatrix} 3 & 0 & 0 & 2m{+}2 & 0 \\ 3 & 0 & 0 & 0 & 2m{+}2 \\ 0 & 3 & 0 & 2{-}m & 0 \\ 0 & 3 & 0 & 0 & 2{-}m \\ 0 & 0 & 3 & 2{-}m & 0 \\ 0 & 0 & 3 & 0 & 2{-}m \end{bmatrix}$$











Laurent-Toric Fugue

& Non-Convex Mirrors

—3D *Proof-of-Concept*—

1611.10300 & 2205.12827
+much more

- (Toric) trans-position:

$$g(y)^{\mathsf{T}} = f(x) = a_1\, x_1^{\,3}\, x_4^{\,2m+2} + a_2\, x_1^{\,3}\, x_5^{\,2m+2} + \underline{a_3}\,\frac{x_2^{\,3}}{x_4^{\,m-2}} + a_4\,\frac{x_3^{\,3}}{x_4^{\,m-2}} + \underline{a_5}\,\frac{x_2^{\,3}}{x_5^{\,m-2}} + a_6\,\frac{x_3^{\,3}}{x_5^{\,m-2}}$$

5×6 matrix of exponents ↕ transpose

$$f(x)^{\mathsf{T}} = g(y) = b_1\, y_1^{\,3}\, y_2^{\,3} + b_2\, \underline{y_3}^{\,3}\, y_4^{\,3} + b_3\, \underline{y_5}^{\,3}\, y_6^{\,3} + b_4\,\frac{y_1^{\,2m+2}}{(\underline{y_3}\,\underline{y_5})^{m-2}} + b_5\,\frac{y_2^{\,2m+2}}{(y_4\, y_6)^{m-2}}$$

deformation

$x_1=1$, $\underline{a_3},\underline{a_5}=0$ $\quad\mathbb{P}^3_{(3:3:1:1)}[8]$

$$a_1\, x_4^{\,8} + a_2\, x_5^{\,8} + a_4\,\frac{x_2^{\,3}}{x_5} + a_6\,\frac{x_3^{\,3}}{x_5} : \left\{\begin{array}{c}(\mathbb{Z}_3{:}\,\tfrac13,\tfrac23,0,0)\\ (\mathbb{Z}_{24}{:}\,\tfrac1{24},\tfrac1{24},0,\tfrac18)\\ \hline (\mathbb{Z}_8{:}\,\tfrac38,\tfrac38,\tfrac18,\tfrac18)\end{array}\right. \begin{bmatrix}x_2\\x_3\\x_4\\x_5\end{bmatrix} : \left\{\begin{array}{l}\mathcal{G}=\mathbb{Z}_3\times\mathbb{Z}_{24},\\ \hline \mathcal{Q}=\mathbb{Z}_8.\end{array}\right.$$

$b_1=0$, $\underline{y_3},\underline{y_5}=1$ $\quad\mathbb{P}^3_{(3:5:8:8)}[24]$

$$b_2\, y_4^{\,3} + b_3\, y_6^{\,3} + b_4\, y_1^{\,8} + b_5\,\frac{y_2^{\,8}}{y_4\, y_6} : \left\{\begin{array}{c}(\mathbb{Z}_8{:}\,\tfrac18,0,0,0)\\ (\mathbb{Z}_3{:}\,0,0,\tfrac13,\tfrac23)\\ \hline (\mathbb{Z}_8{:}\,\tfrac5{24},\tfrac3{24},\tfrac13,\tfrac13)\end{array}\right. \begin{bmatrix}y_1\\y_2\\y_4\\y_6\end{bmatrix} : \left\{\begin{array}{l}\mathcal{G}^\nabla=\mathbb{Z}_8\times\mathbb{Z}_3\\ \hline \mathcal{Q}^\nabla=\mathbb{Z}_{24}.\end{array}\right.$$

quotient ***either*** one of the two by $\mathbb{Z}_3$

$x_1=1$, $a_4,a_5=0$ $\quad\mathbb{P}^3_{(3:3:1:1)}[8]$

$$a_1\, x_4^{\,8} + a_2\, x_5^{\,8} + a_4\,\frac{x_2^{\,3}}{x_5} + a_5\,\frac{x_3^{\,3}}{x_4} : \left\{\begin{array}{c}(\mathbb{Z}_3{:}\,\tfrac13,\tfrac13,0,0)\\ (\mathbb{Z}_{24}{:}\,\tfrac1{24},\tfrac{23}{24},\tfrac18,\tfrac78)\\ \hline (\mathbb{Z}_8{:}\,\tfrac38,\tfrac38,\tfrac18,\tfrac18)\end{array}\right. \begin{bmatrix}x_2\\x_3\\x_4\\x_5\end{bmatrix} : \left\{\begin{array}{l}\mathcal{G}=\mathbb{Z}_3\times\mathbb{Z}_6\\ \hline \mathcal{Q}=\mathbb{Z}_8\times\mathbb{Z}_4\end{array}\right.$$

$/\mathbb{Z}_4$

$b_1=0$, $y_4,y_5=1$ $\quad\mathbb{P}^3_{(1:1:2:2)}[6]$

$$b_2\, y_4^{\,3} + b_3\, y_5^{\,3} + b_4\,\frac{y_1^{\,8}}{y_5} + b_5\,\frac{y_2^{\,8}}{y_4} : \left\{\begin{array}{c}(\mathbb{Z}_4{:}\,\tfrac14,\tfrac14,0,0)\\ (\mathbb{Z}_{24}{:}\,\tfrac1{24},\tfrac{23}{24},\tfrac13,\tfrac23)\\ \hline (\mathbb{Z}_6{:}\,\tfrac16,\tfrac16,\tfrac13,\tfrac13)\end{array}\right. \begin{bmatrix}y_1\\y_2\\y_3\\y_6\end{bmatrix} : \left\{\begin{array}{l}\mathcal{G}^\nabla=\mathbb{Z}_4\times\mathbb{Z}_8\\ \hline \mathcal{Q}^\nabla=\mathbb{Z}_6\times\mathbb{Z}_3\end{array}\right.$$

for example

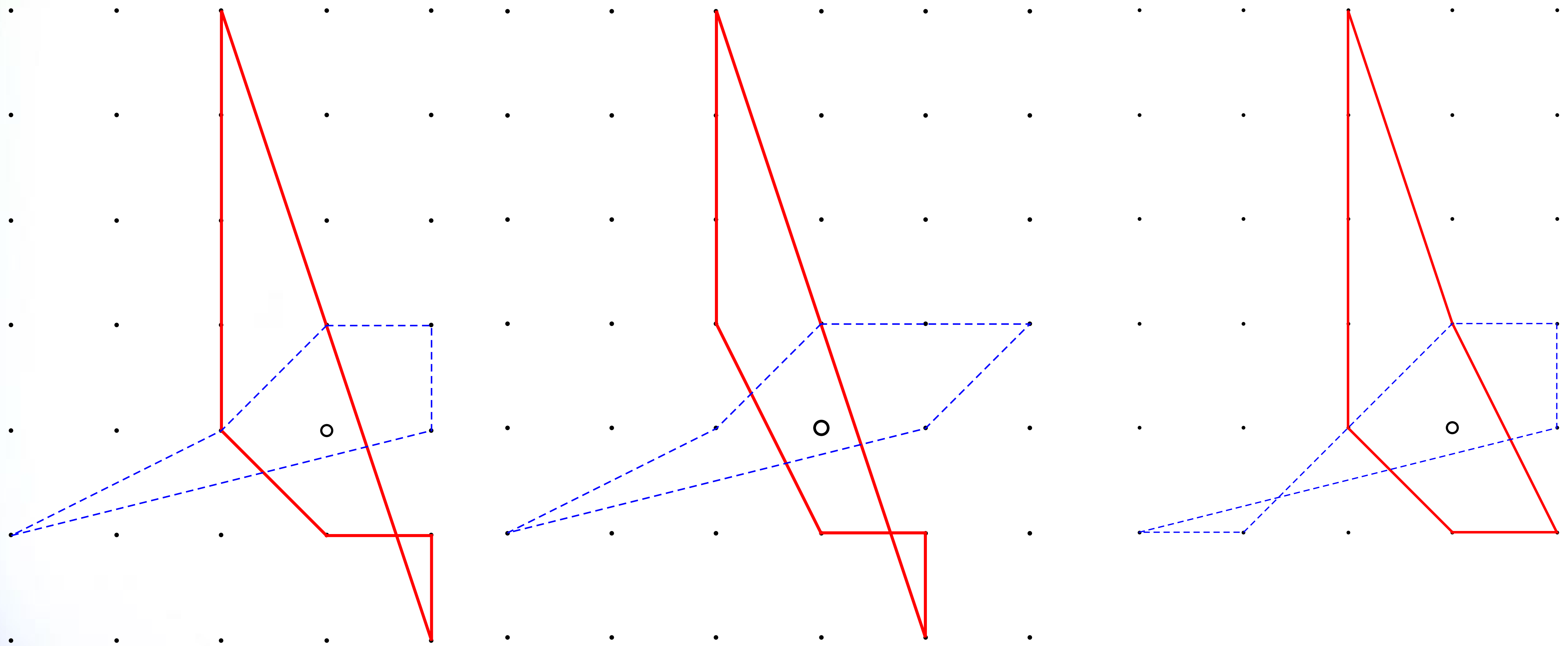
$/\mathbb{Z}_3$

Laurent-Toric Fugue

& Non-Convex Mirrors

—*Proof-of-Concept*—

- Not just Hirzebruch scrolls, either:
  - Buckets of 2-dimensional polygons, invented to test $\nabla : \Delta^{\star} \xleftrightarrow{1\text{–}1} \Delta$



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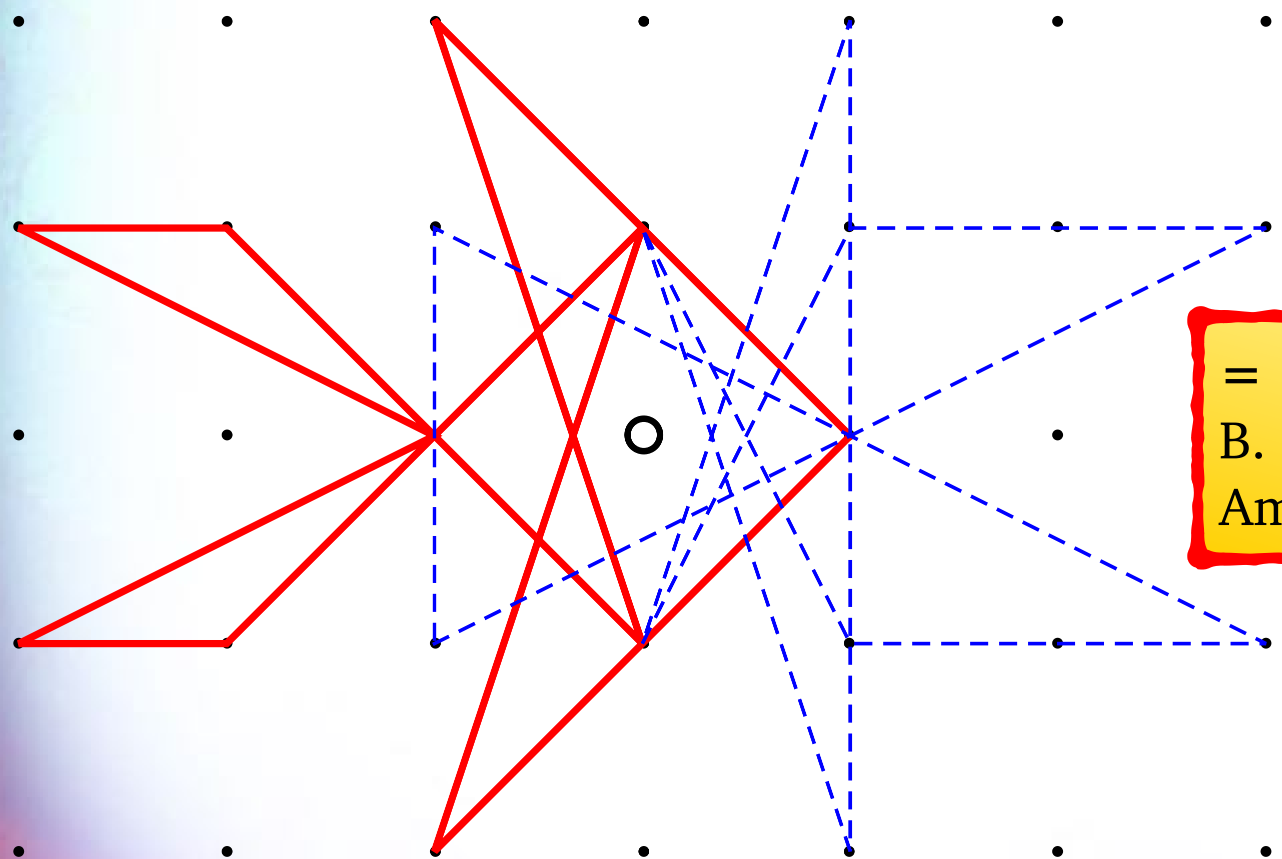
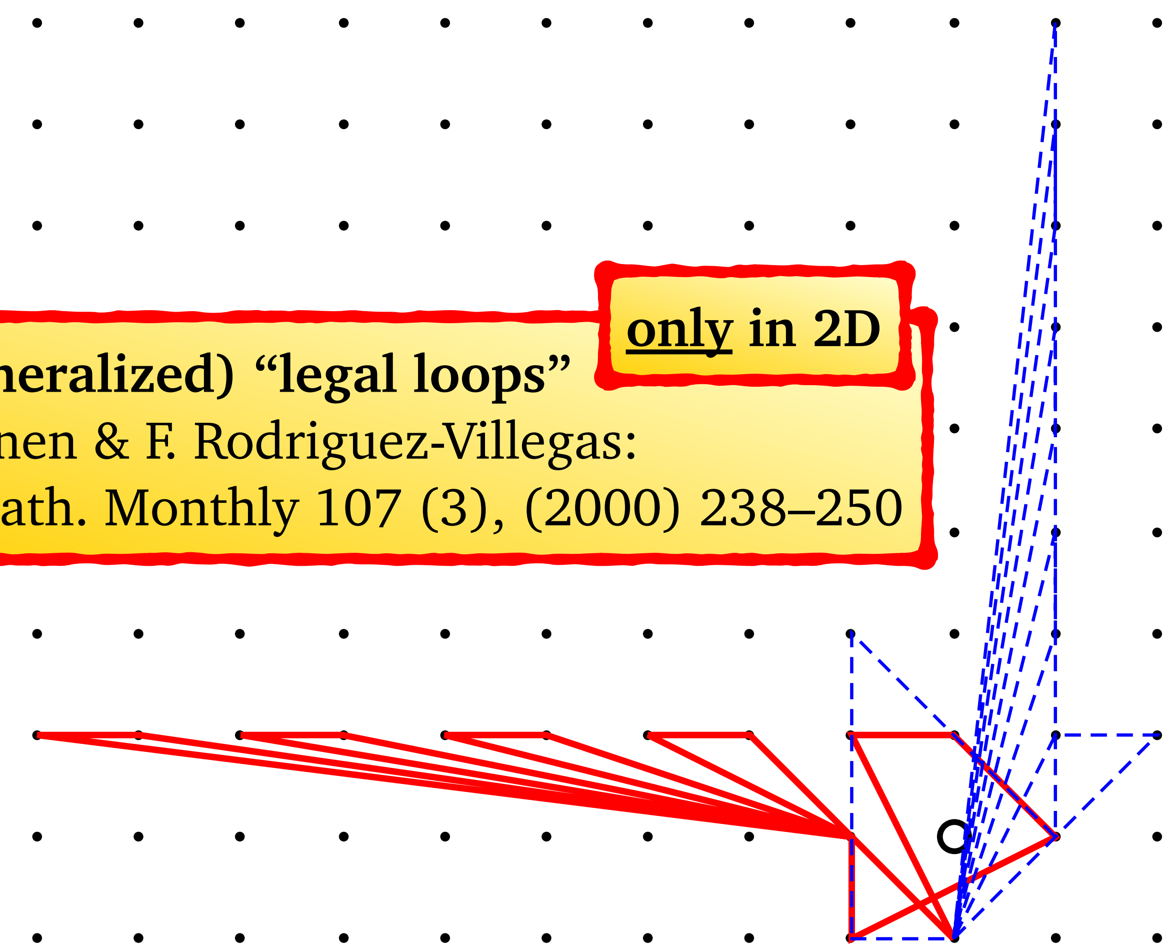




only in 2D

= (generalized) "legal loops"
B. Poonen & F. Rodriguez-Villegas:
Am. Math. Monthly 107 (3), (2000) 238–250

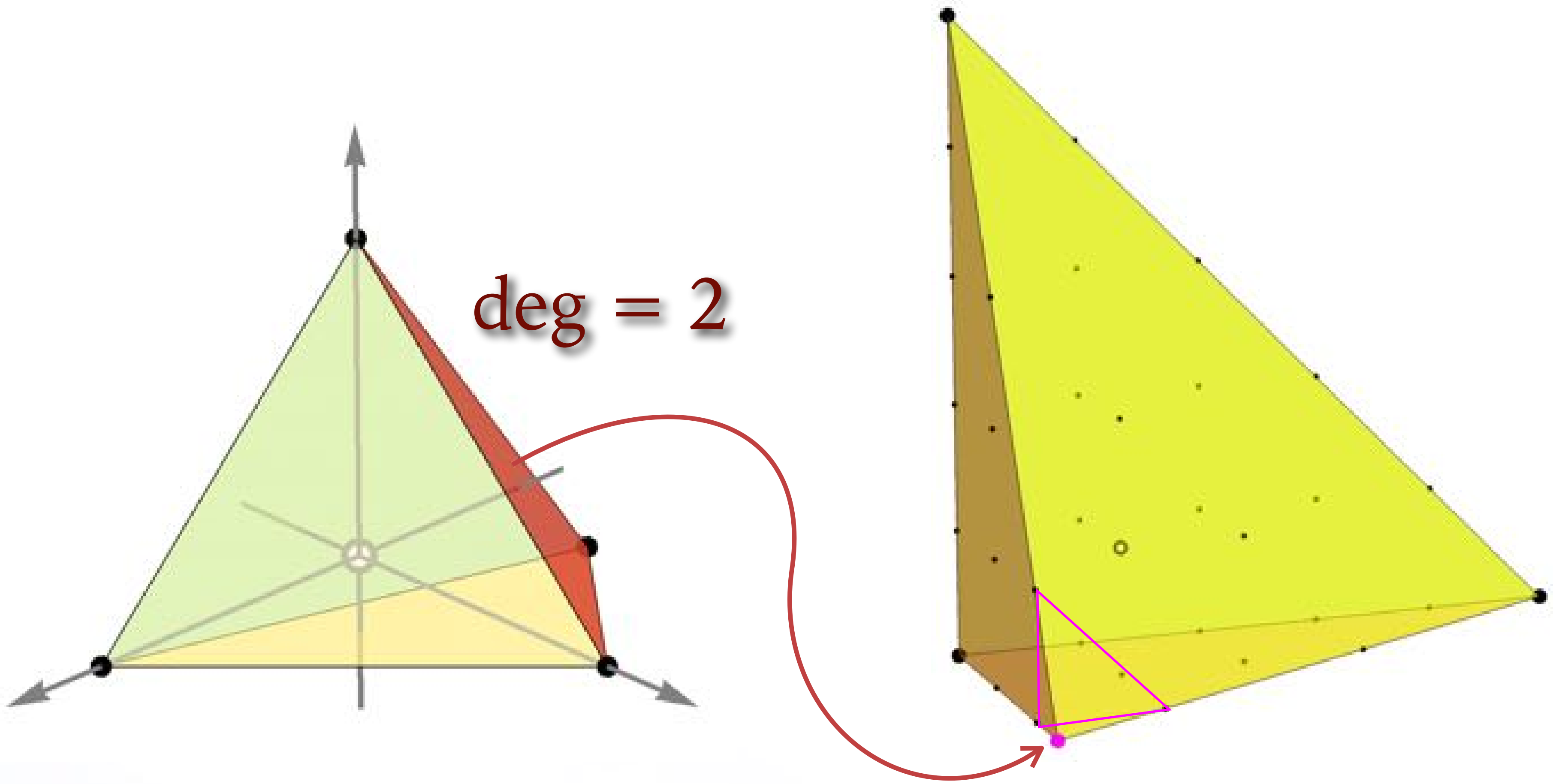
Laurent-Toric Fugue

& Non-Convex Mirrors

—*Proof-of-Concept*—

1611.10300 & 2205.12827
+much more

- Not just Hirzebruch scrolls, either:
  - Buckets of 2-dimensional polygons, invented to test $\nabla : \Delta^\star \overset{1\text{–}1}{\longleftrightarrow} \Delta$
  - And, plenty of 3-dimensional polyhedra:



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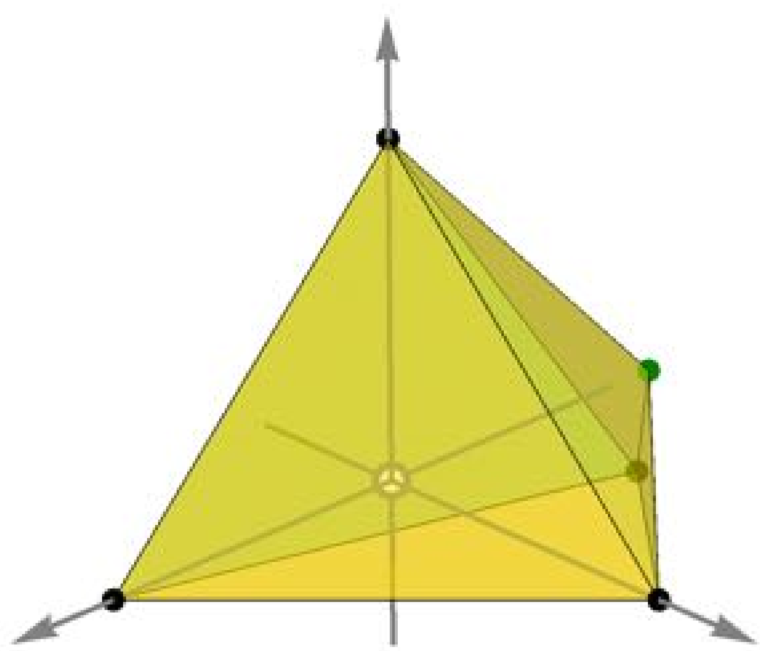

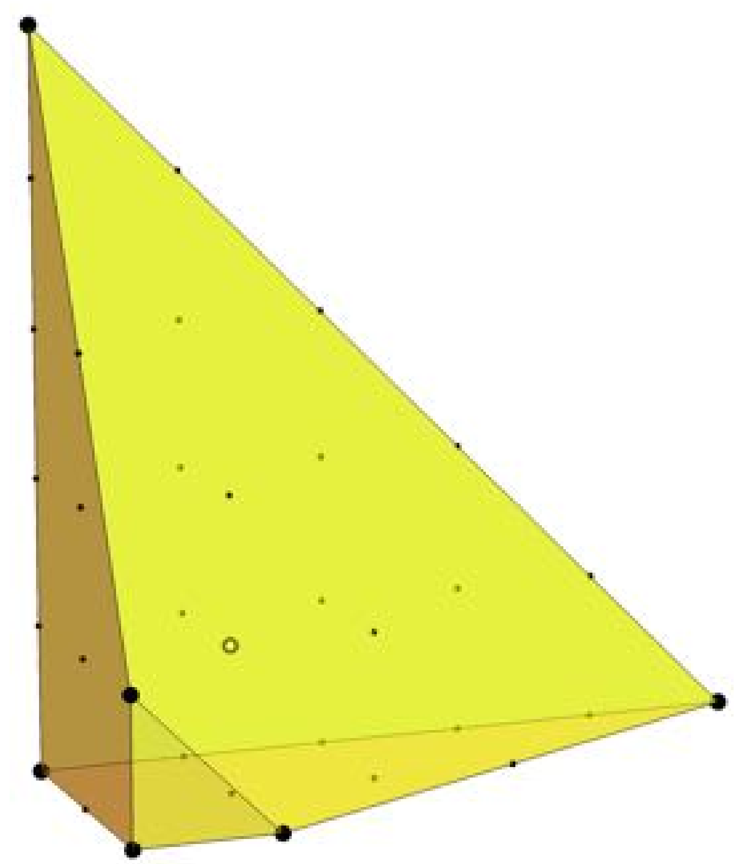

Laurent-Toric Fugue

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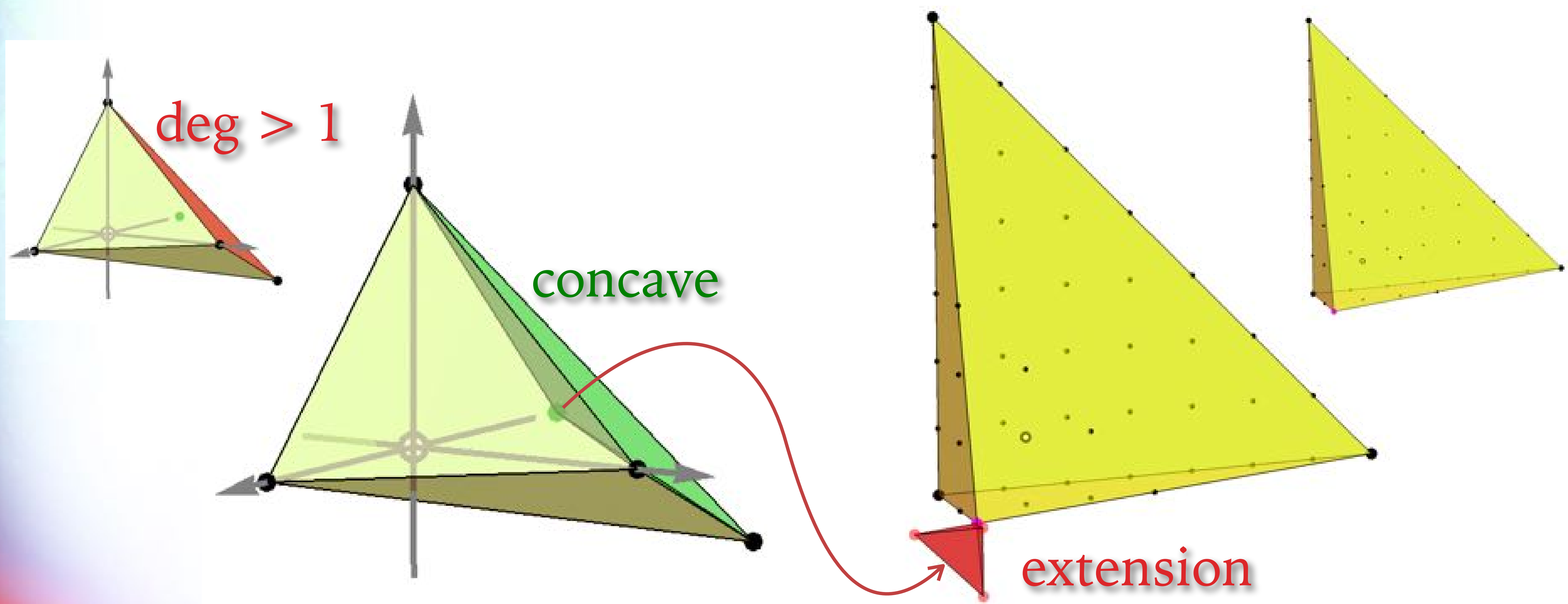

Laurent-Toric Fugue

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+much more

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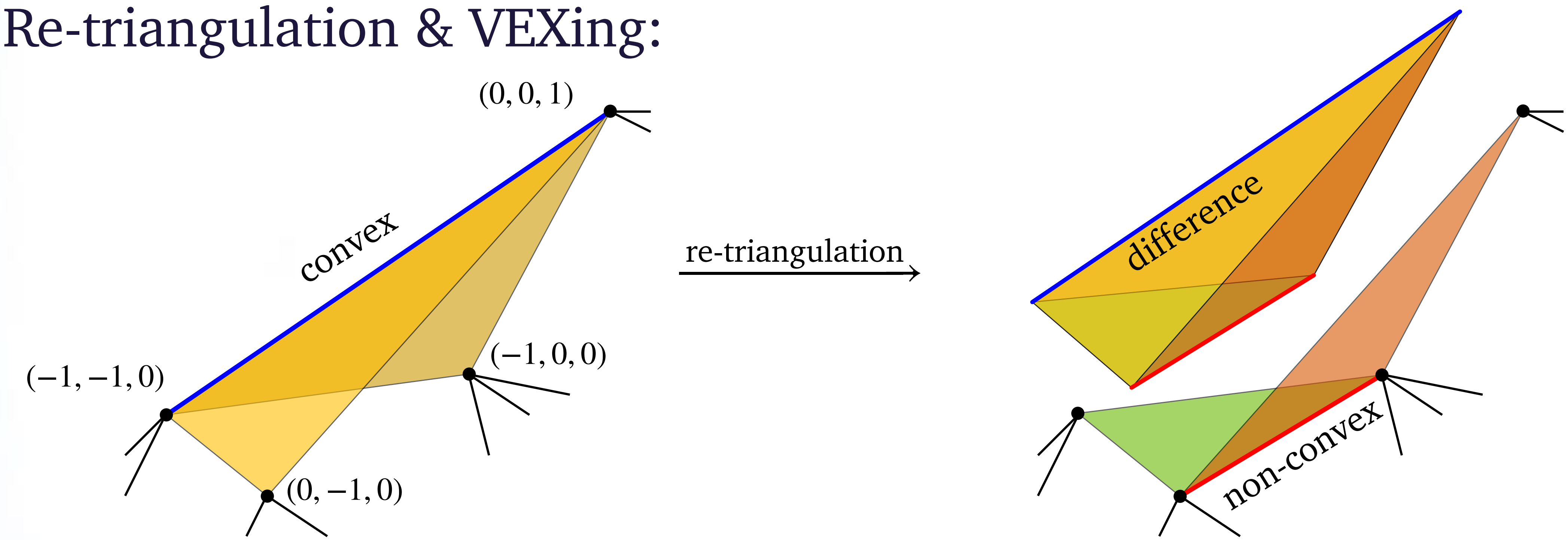
  - And, plenty of 3-dimensional polyhedra:
  - Re-triangulation & VEXing:



Laurent-Toric Fugue

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—*Proof-of-Concept*—

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+much more

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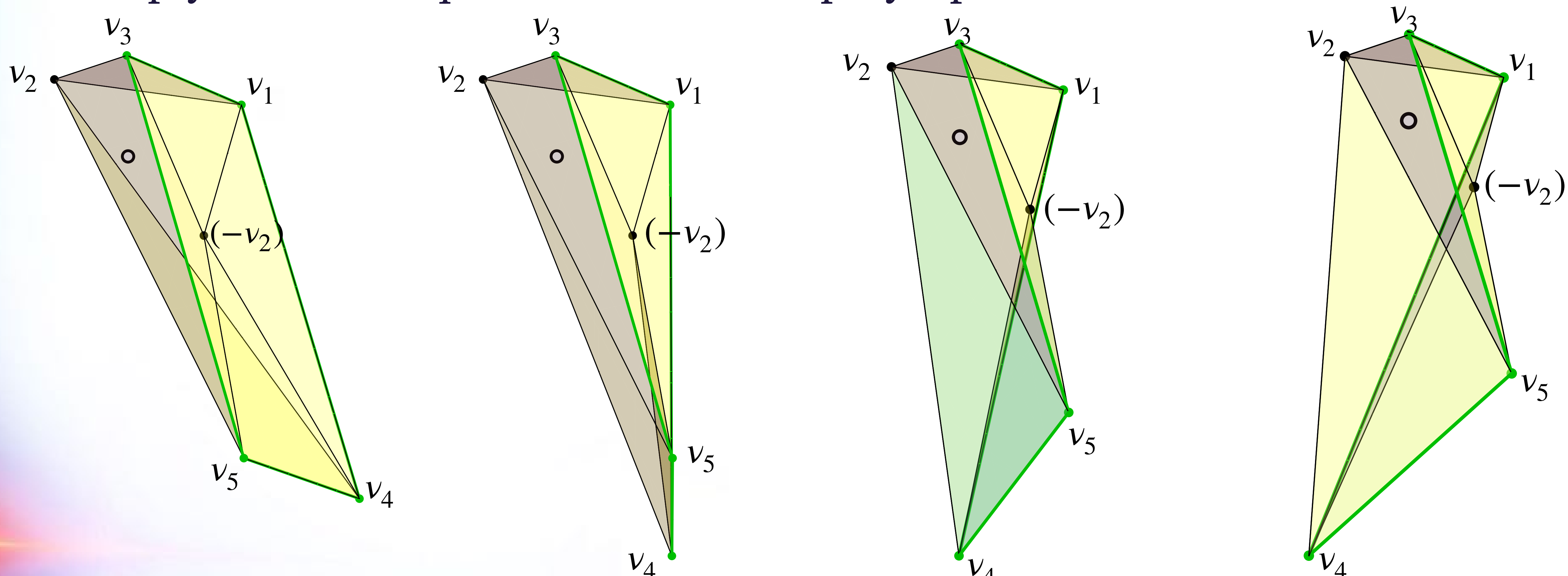
  - And, plenty of 3-dimensional polyhedra:
  - Re-triangulation & VEXing:
  - Multiply infinite sequences of twisted polytopes:



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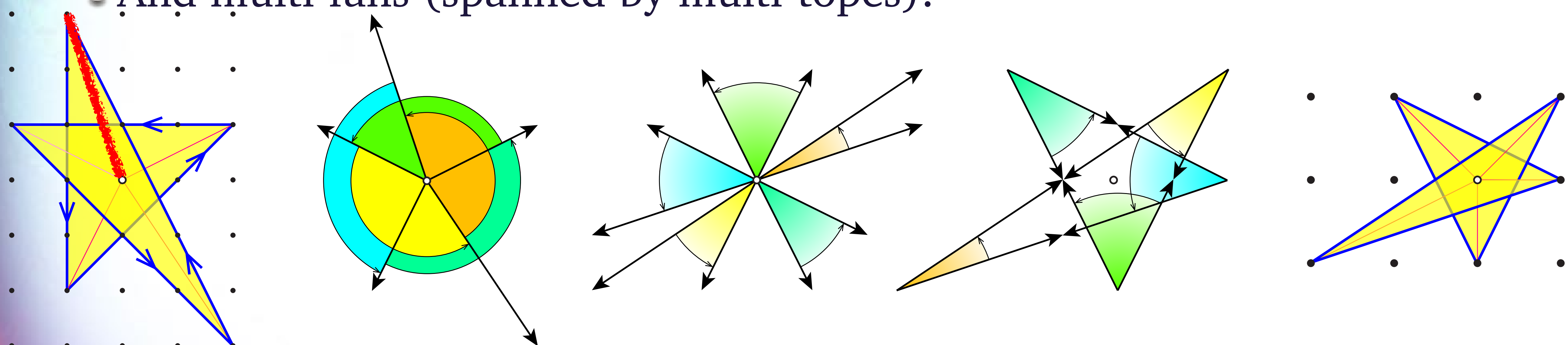
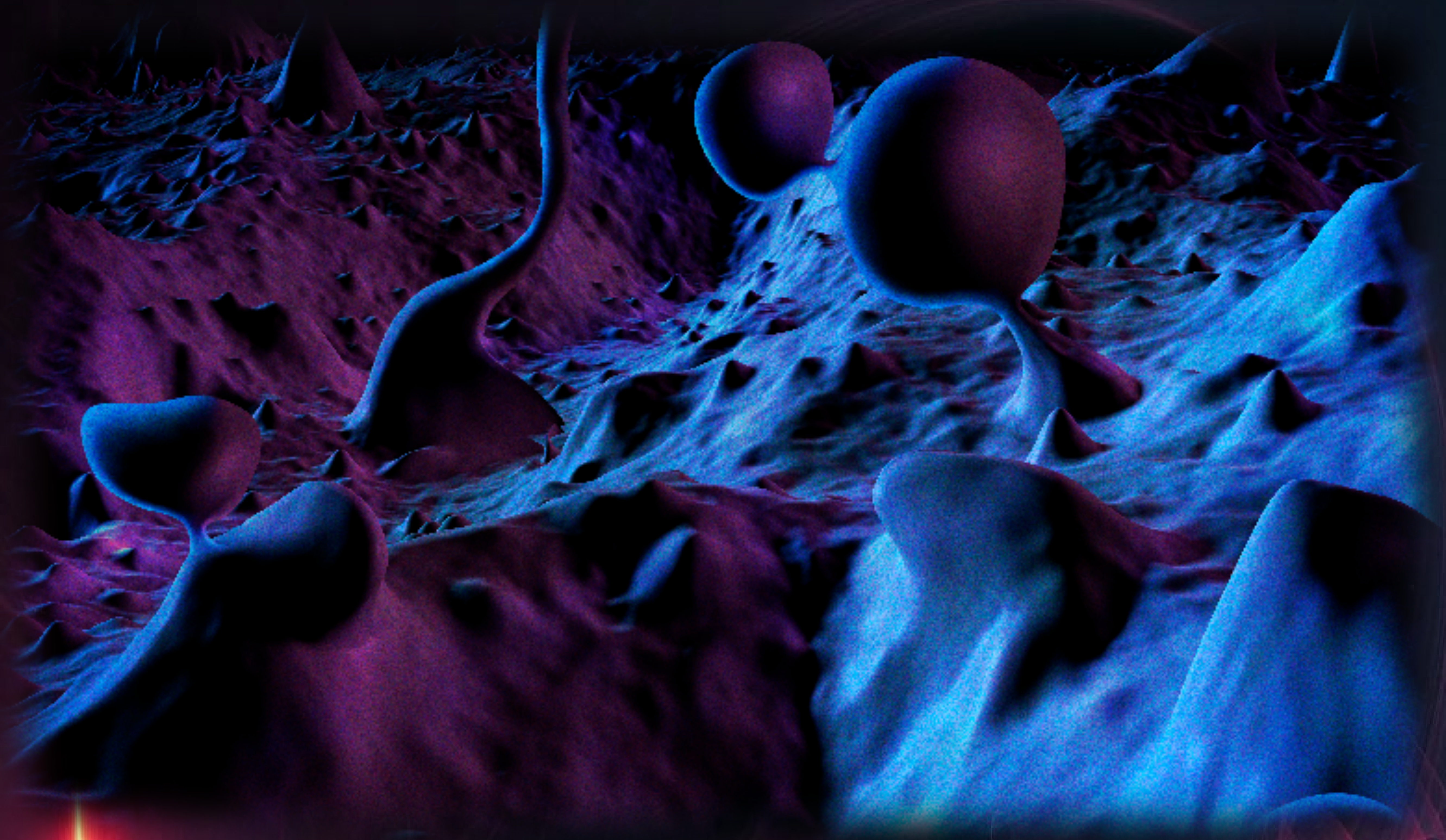
  - And, plenty of 3-dimensional polyhedra:
  - Re-triangulation & VEXing:
  - Multiply infinite sequences of twisted polytopes:
  - And multi-fans (spanned by multi-topes):



winding number (multiplicity, Duistermaat-Heckman fn.) = 2 $= \chi(\mathcal{O})$

[A. Hattori+M. Masuda" *Theory of Multi-Fans*, Osaka J. Math. **40** (2003) 1–68]



Discriminant Divertimento

(How Small Can We Go?)

Discriminant Divertimento

1611.10300 & 2205.12827 +much more

The Phase-Space = 2nd Fan

—*Proof-of-Concept*—

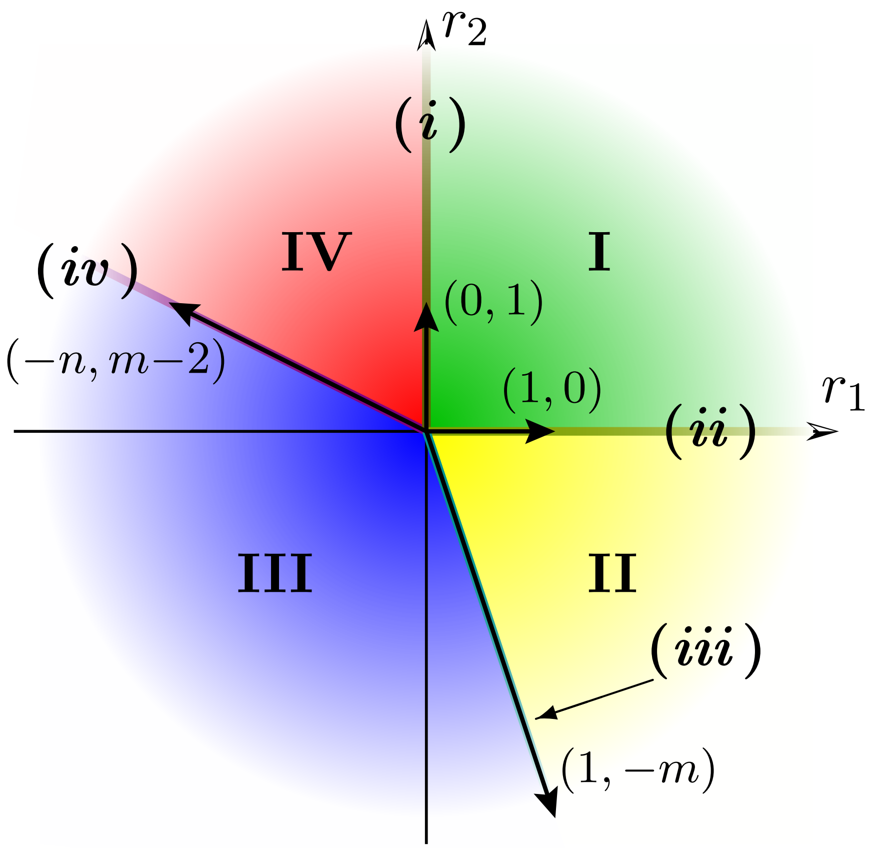
- The (super)potential: $W(X) := X_0 \cdot f(X)$,

$$f(X) := \sum_{j=1}^{2} \left(\sum_{i=2}^{n} (a_{ij}\, X_i^n) X_{n+j}^{2-m} + a_j\, X_1^n X_{n+j}^{(n-1)m+2} \right)$$

	X_0	X_1	X_2	$\cdots$	X_n	X_{n+1}	X_{n+2}
Q^1	$-n$	1	1	$\cdots$	1	0	0
Q^2	$m-2$	$-m$	0	$\cdots$	0	1	1

- The possible vevs

	$\|x_0\|$	$\|x_1\|$	$\|x_2\|$	$\cdots$	$\|x_n\|$	$\|x_{n+1}\|$	$\|x_{n+2}\|$
i	0	0	0	$\cdots$	0	$*$	$*$
I	0	$*$	$*$	$\cdots$	$*$	$*$	$*$
ii	0	0	$*$	$\cdots$	$*$	0	0
II	0	$\|x_1\| = \sqrt{\frac{\sum_j \|x_{n+j}\|^2 - r_2}{m}} = \sqrt{r_1 - \sum_{i=2}^{n} \|x_i\|^2} > 0$	$*$	$\cdots$	$*$	$*$	$*$
iii	0	$\sqrt{r_1}$	0	$\cdots$	0	0	0
III	$\sqrt{\frac{mr_1+r_2}{(n-1)m+2}}$	$\sqrt{\frac{(m-2)r_1+nr_2}{(n-1)m+2}}$	0	$\cdots$	0	0	0
iv	$\sqrt{-r_1/n}$	0	0	$\cdots$	0	0	0
IV	$\sqrt{-r_1/n}$	0	0	$\cdots$	0	$*$	$*$



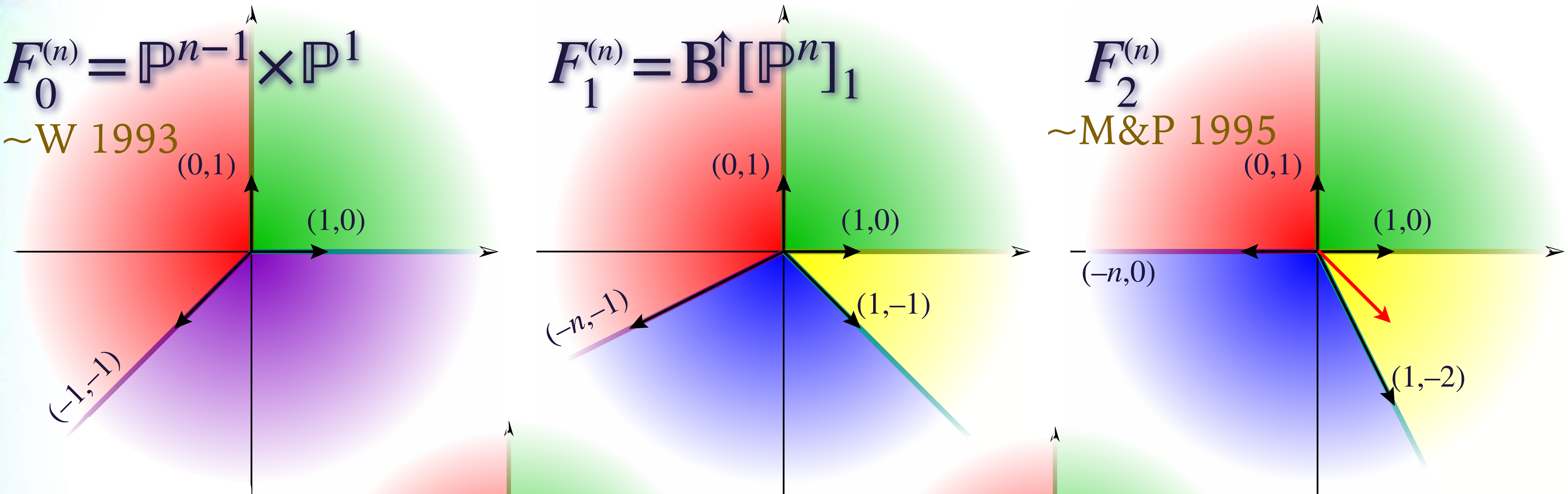
Discriminant Divertimento

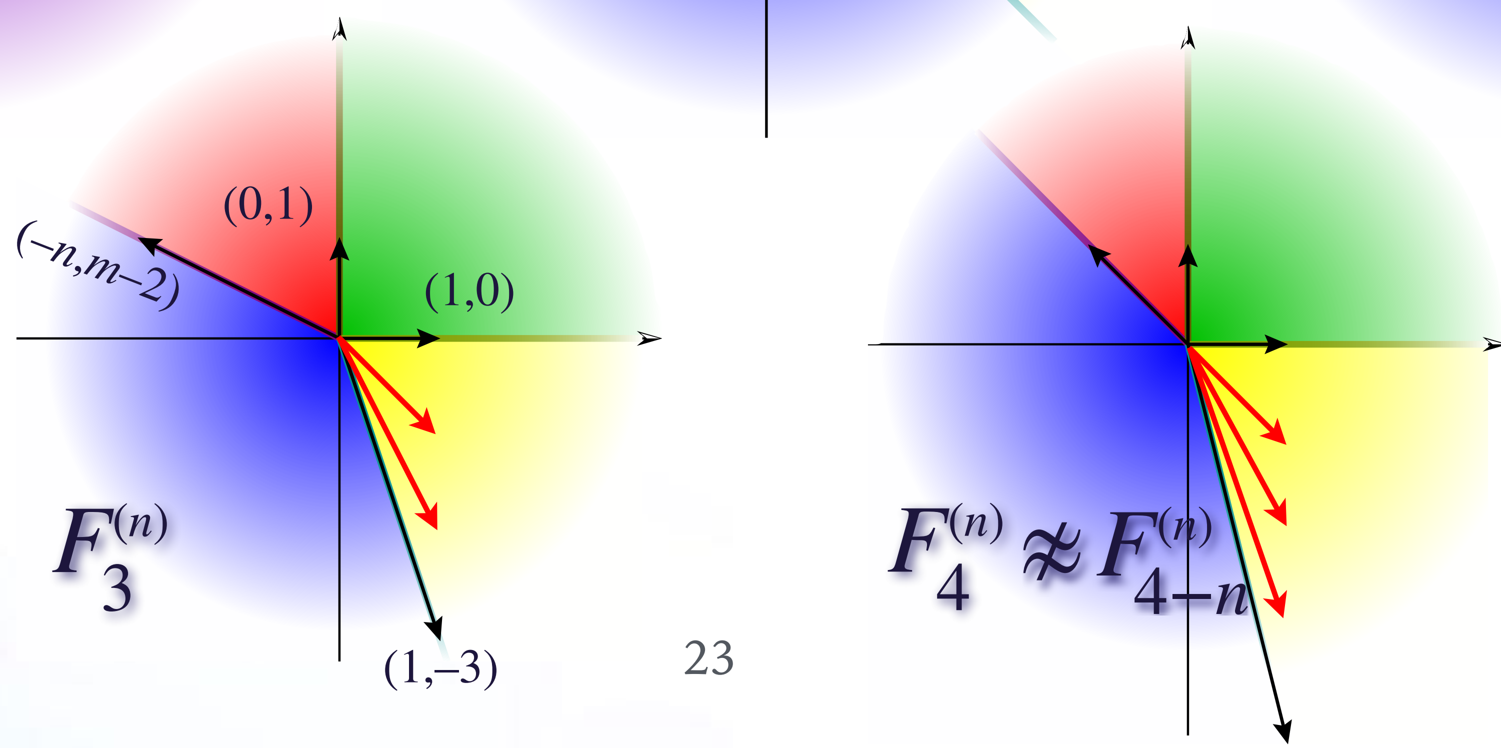
1611.10300 & 2205.12827 +much more

The Phase-Space = 2nd Fan

—Proof-of-Concept—

- Varying m in $F^{(n)}_m$:





• • • • • •

Secondary fans are all different

Discriminant Divertimento

The *A*-Discriminant

—*Proof-of-Concept*—

- Now add worldsheet instantons:
  - Near $(r_1, r_2) = (0,0)$, classical analysis of Kähler (metric) phase-space fails [M&P: arXiv:hep-th/9412236]
- With

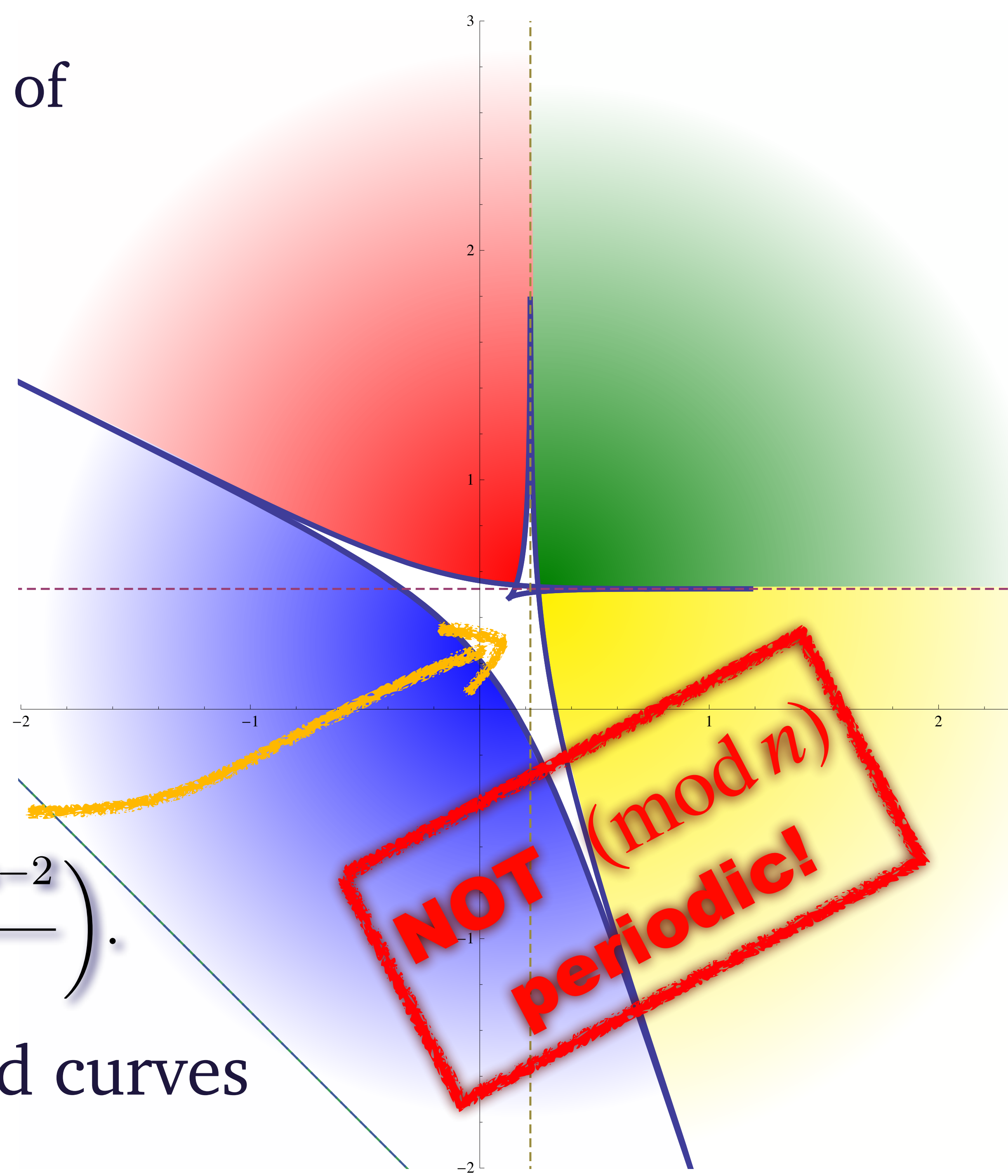
	X_0	X_1	X_2	$\cdots$	X_n	X_{n+1}	X_{n+2}
Q^1	$-n$	1	1	$\cdots$	1	0	0
Q^2	$m-2$	$-m$	0	$\cdots$	0	1	1

- the instanton resummation gives:

$$r_1 + \frac{\hat{\theta}_1}{2\pi i} = -\frac{1}{2\pi}\log\left(\frac{\sigma_1^{n-1}\,(\sigma_1 - m\,\sigma_2)}{[(m-2)\sigma_2 - n\sigma_1]^n}\right),$$

$$r_2 + \frac{\hat{\theta}_2}{2\pi i} = -\frac{1}{2\pi}\log\left(\frac{\sigma_2^2\,[(m-2)\sigma_2 - n\sigma_1]^{m-2}}{(\sigma_1 - m\,\sigma_2)^m}\right).$$

a cumulative measure of embedded curves



Discriminant Divertimento

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- the instanton resummation gives:

$$r_1 + \frac{\hat{\theta}_1}{2\pi i} = -\frac{1}{2\pi} \log\left(\frac{\sigma_1^{\,n-1}\,(\sigma_1 - m\,\sigma_2)}{[(m-2)\sigma_2 - n\sigma_1]^n}\right),$$

$$r_2 + \frac{\hat{\theta}_2}{2\pi i} = -\frac{1}{2\pi} \log\left(\frac{\sigma_2^2\,[(m-2)\sigma_2 - n\sigma_1]^{m-2}}{(\sigma_1 - m\,\sigma_2)^m}\right).$$

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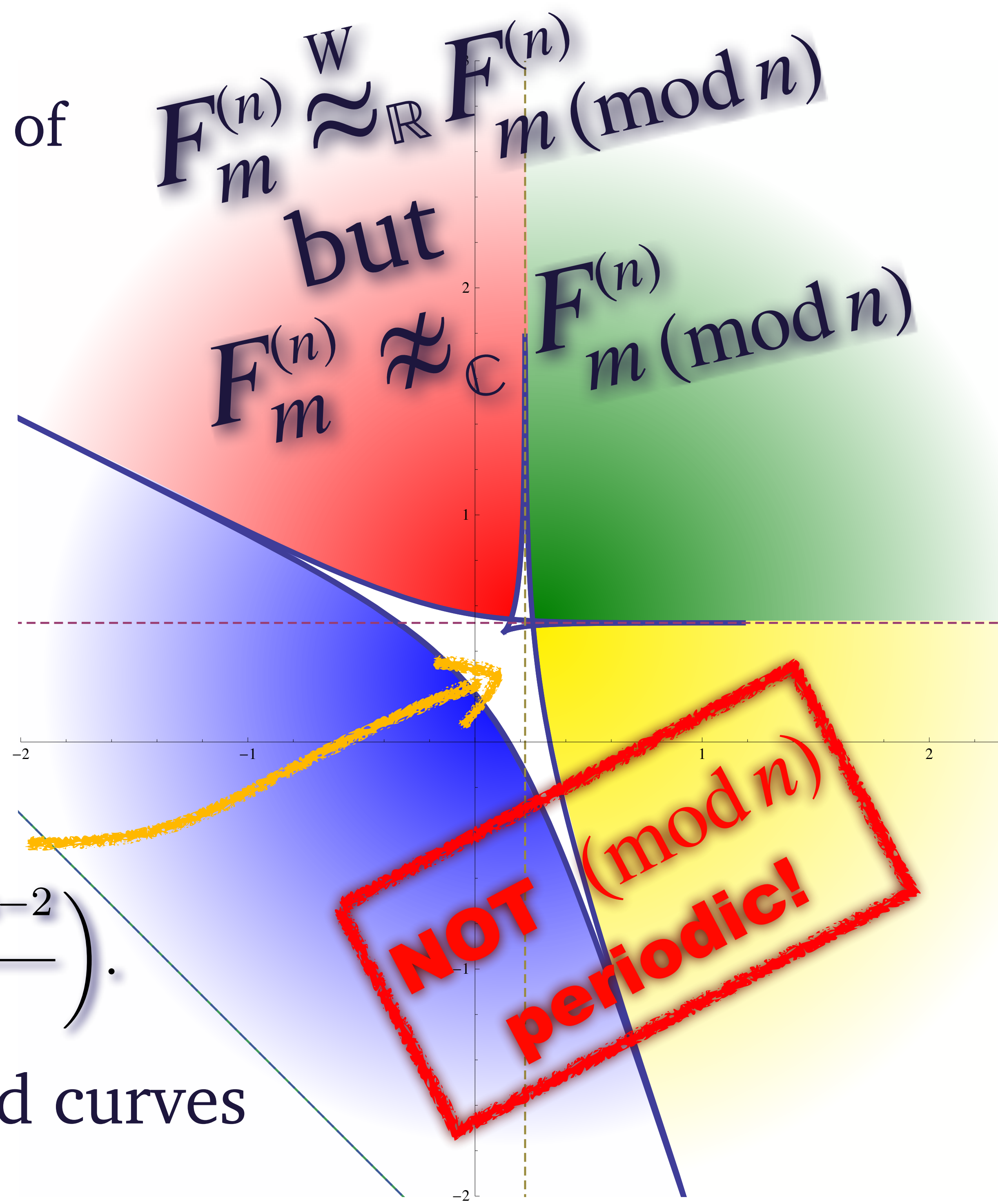


…and a Mirror Motet

(Yes, *the* B^3H^2K-mirrors)

Mirror Motets

The A-Discriminant

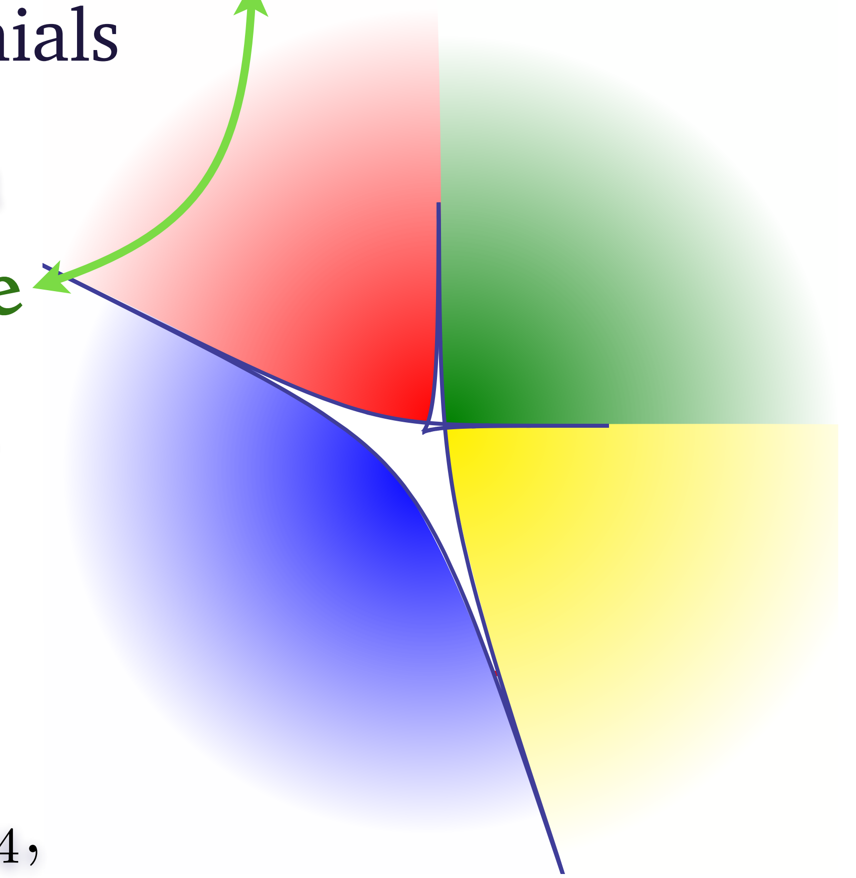
—*Proof-of-Concept*—

- Now compare with the complex structure of the B^3H^2K-mirror
  - Restricted to the "cornerstone" defining polynomials

$$f(x) = a_0 \prod_{\nu_i\in\Delta^\star} (x_{\nu_i})^{\langle \nu_i,\mu_0\rangle+1} + \sum_{\mu_I\in\Delta} a_{\mu_I} \prod_{\nu_i\in\Delta^\star}(x_{\nu_i})^{\langle\nu_i,\mu_I\rangle+1}$$

$$g(y) = b_0 \prod_{\mu_I\in\Delta}(y_{\mu_I})^{\langle\mu_I,\nu_0\rangle+1} + \sum_{\nu_i\in\Delta^\star} b_{\nu_i}\prod_{\mu_I\in\Delta}(y_{\mu_I})^{\langle\mu_I,\nu_i\rangle+1}$$

transpose



- In particular,

$$g(y) = \sum_{i=0}^{n+2} b_i\,\phi_i(y) = b_0\,\phi_0 + b_1\,\phi_1 + b_2\,\phi_2 + b_3\,\phi_3 + b_4\,\phi_4,$$

$$\phi_0 := y_1\cdots y_4,\quad \phi_1 := y_1^2\,y_2^2,\quad \phi_2 := y_3^2\,y_4^2,\quad \phi_3 := \frac{y_1^{m+2}}{y_3^{m-2}},\quad \phi_4 := \frac{y_2^{m+2}}{y_4^{m-2}},$$

$$z_1 = -\frac{\beta\,[(m-2)\beta+m]}{m+2},\qquad z_2 = \frac{(2\beta+1)^2}{(m+2)^2\,\beta^m},\qquad \beta := \left[\frac{b_1\,\phi_1}{b_0\,\phi_0}\Big/{}^{A}\mathscr{J}(g)\right],$$

$\phi_0^2 = \phi_1\phi_2$ etc.

Mirror Motets

The A-Discriminant

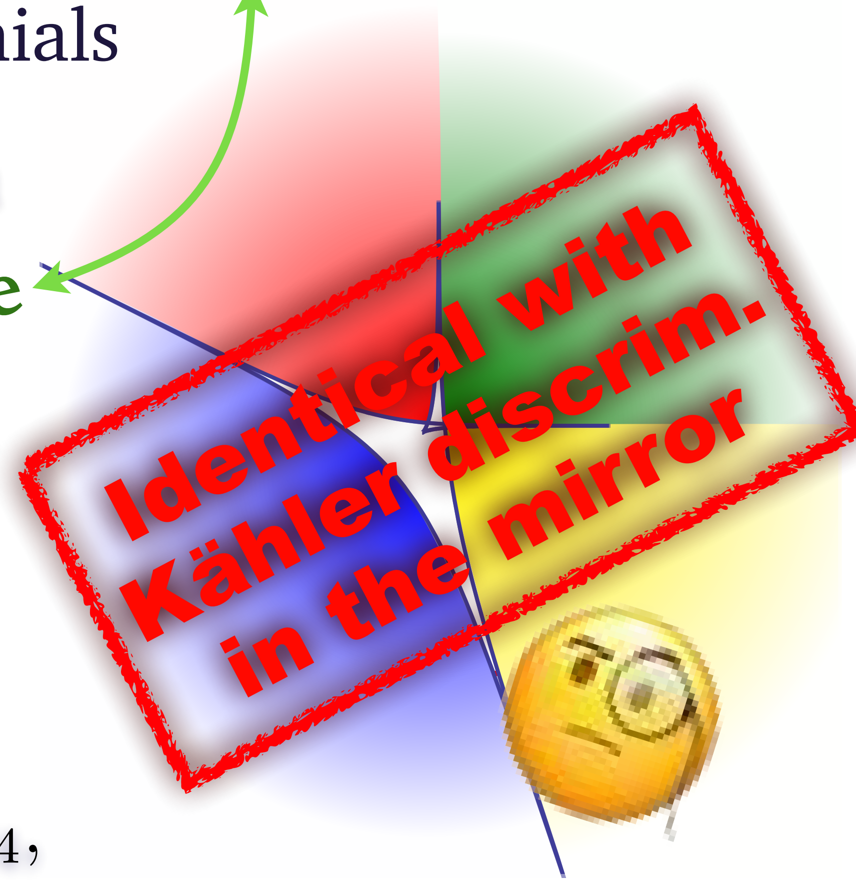
—Proof-of-Concept—

- Now compare with the complex structure of the B³H²K-mirror
  - Restricted to the "cornerstone" defining polynomials

$$f(x) = a_0 \prod_{\nu_i\in\Delta^\star} (x_{\nu_i})^{\langle\nu_i,\mu_0\rangle+1} + \sum_{\mu_I\in\Delta} a_{\mu_I} \prod_{\nu_i\in\Delta^\star} (x_{\nu_i})^{\langle\nu_i,\mu_I\rangle+1}$$

transpose

$$g(y) = b_0 \prod_{\mu_I\in\Delta} (y_{\mu_I})^{\langle\mu_I,\nu_0\rangle+1} + \sum_{\nu_i\in\Delta^\star} b_{\nu_i} \prod_{\mu_I\in\Delta} (y_{\mu_I})^{\langle\mu_I,\nu_i\rangle+1}$$



- In particular,

$$g(y) = \sum_{i=0}^{n+2} b_i\,\phi_i(y) \;= b_0\,\phi_0 + b_1\,\phi_1 + b_2\,\phi_2 + b_3\,\phi_3 + b_4\,\phi_4,$$

$$\phi_0 := y_1\cdots y_4,\quad \phi_1 := y_1^2\,y_2^2,\quad \phi_2 := y_3^2\,y_4^2,\quad \phi_3 := \frac{y_1^{m+2}}{y_3^{m-2}},\quad \phi_4 := \frac{y_2^{m+2}}{y_4^{m-2}},$$

$$z_1 = -\frac{\beta\,[(m-2)\beta+m]}{m+2},\qquad z_2 = \frac{(2\beta+1)^2}{(m+2)^2\,\beta^m},\qquad \beta := \left[\frac{b_1\,\phi_1}{b_0\,\phi_0}\Big/{}^{A}\mathscr{J}(g)\right],$$

$\phi_0^2 = \phi_1\phi_2$ etc.

Mirror Motets

The A-Discriminant

—Proof-of-Concept—

- So: $\mathscr{M}({}^\nabla F^{(n)}_m[c_1]) \overset{\text{mm}}{\approx} \mathscr{W}(F^{(n)}_m[c_1])$ — easy: 2-dimensional
- In fact, also: $\mathscr{W}({}^\nabla F^{(n)}_m[c_1]) \overset{\text{mm}}{\approx} \mathscr{M}(F^{(n)}_m[c_1])$
- ✅...restricted to no (MPCP) blow-ups; only "cornerstone" polynomials
- Then, $\dim \mathscr{W}({}^\nabla F^{(n)}_m[c_1]) = n = \dim \mathscr{M}(F^{(n)}_m[c_1])$
- Same methods:

$$e^{2\pi i\,\widetilde{\tau}_\alpha} = \prod_{I=0}^{2n}\Big(\sum_{\beta=1}^{2}\widetilde{Q}_I^\beta\,\widetilde{\sigma}_\beta\Big)^{\widetilde{Q}_I^\alpha}$$

$$\tilde{z}_a = \prod_{I=0}^{2n}(a_I\,\varphi_I(x))^{\widetilde{Q}_I^\alpha}\Big/{}^A\mathscr{J}$$

I	$\left(\sum_\beta \widetilde{Q}_I^\beta\,\widetilde{\sigma}_\beta\right)$	$n=4$ $(a_I\,\varphi_I)/{}^A\mathscr{J}_{(210)}(f)$
0	$-2(m+2)(\widetilde{\sigma}_1+\widetilde{\sigma}_2)$	$-2\big((a_3\,\varphi_3)+(a_4\,\varphi_4)\big)$
1	$m\,\widetilde{\sigma}_1+2\,\widetilde{\sigma}_2$	$\frac{m\,(a_3\,\varphi_3)+2\,(a_4\,\varphi_4)}{m+2}$
2	$2\,\widetilde{\sigma}_1+m\,\widetilde{\sigma}_2$	$\frac{2\,(a_3\,\varphi_3)+m\,(a_4\,\varphi_4)}{m+2}$
3	$(m+2)\,\widetilde{\sigma}_1$	$(a_3\,\varphi_3)$
4	$(m+2)\,\widetilde{\sigma}_2$	$(a_4\,\varphi_4)$

Mirror Motets

The A-Discriminant

—*Proof-of-Concept*—

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- Same methods:

$$e^{2\pi i\,\tilde\tau_\alpha} = \prod_{I=0}^{2n}\Big(\sum_{\beta=1}^{2}\tilde Q^\beta_I\,\tilde\sigma_\beta\Big)^{\tilde Q^\alpha_I}$$

$$\tilde z_a = \prod_{I=0}^{2n}\big(a_I\,\varphi_I(x)\big)^{\tilde Q^\alpha_I}\Big/{}^A\mathscr{J}$$

I	Kähler: $\big(\sum_\beta \tilde Q^\beta_I\,\tilde\sigma_\beta\big)$	$n=4$ complex structure: $(a_I\,\varphi_I)/{}^A\mathscr{J}_{(210)}(f)$
0	$-2(m{+}2)(\tilde\sigma_1+\tilde\sigma_2)$	$-2\big((a_3\,\varphi_3)+(a_4\,\varphi_4)\big)$
1	$m\,\tilde\sigma_1+2\,\tilde\sigma_2$	$\frac{m\,(a_3\,\varphi_3)+2\,(a_4\,\varphi_4)}{m+2}$
2	$2\,\tilde\sigma_1+m\,\tilde\sigma_2$	$\frac{2\,(a_3\,\varphi_3)+m\,(a_4\,\varphi_4)}{m+2}$
3	$(m{+}2)\,\tilde\sigma_1$	$(a_3\,\varphi_3)$
4	$(m{+}2)\,\tilde\sigma_2$	$(a_4\,\varphi_4)$

$(m{+}2)\tilde\sigma_i \mapsto a_{i+2}\varphi_{i+2}$

Laurent GLSM Coda

arXiv: "real soon"

Summary

—*Proof-of-Concept*—

- CY($n-1$)-folds in Hirzebruch n-folds
  - Euler characteristic ✅
  - Chern class, term-by-term ✅
  - Hodge numbers ✅ (*jump @* $^\sharp\mathscr{X}$)
  - Cornerstone polynomials & mirror ✅
  - Phase-space regions & mirror ✅
  - Phase-space discriminant & mirror ✅
  - The "other way around" ✅ (*limited!*)
  - Yukawa couplings ✅
  - World-sheet instantons ✅
  - Gromov-Witten invariants SOON ?✅
    - *Will there be anything else? …being ML-datamined*

- Oriented polytopes
- Trans-polar$^\triangledown$ constr.
- Newton $\Delta_X := (\Delta^\star_X)^\triangledown$
- VEX polytopes
  s.t.: $((\Delta)^\triangledown)^\triangledown = \Delta$
- Star-triangulable
  w/flip-folded faces
- Polytope *extension*
  ⇔ Laurent monomials

B^3H^2K mirrors

& GLSM
Toric textbooks to be
😇😈 *…extended*

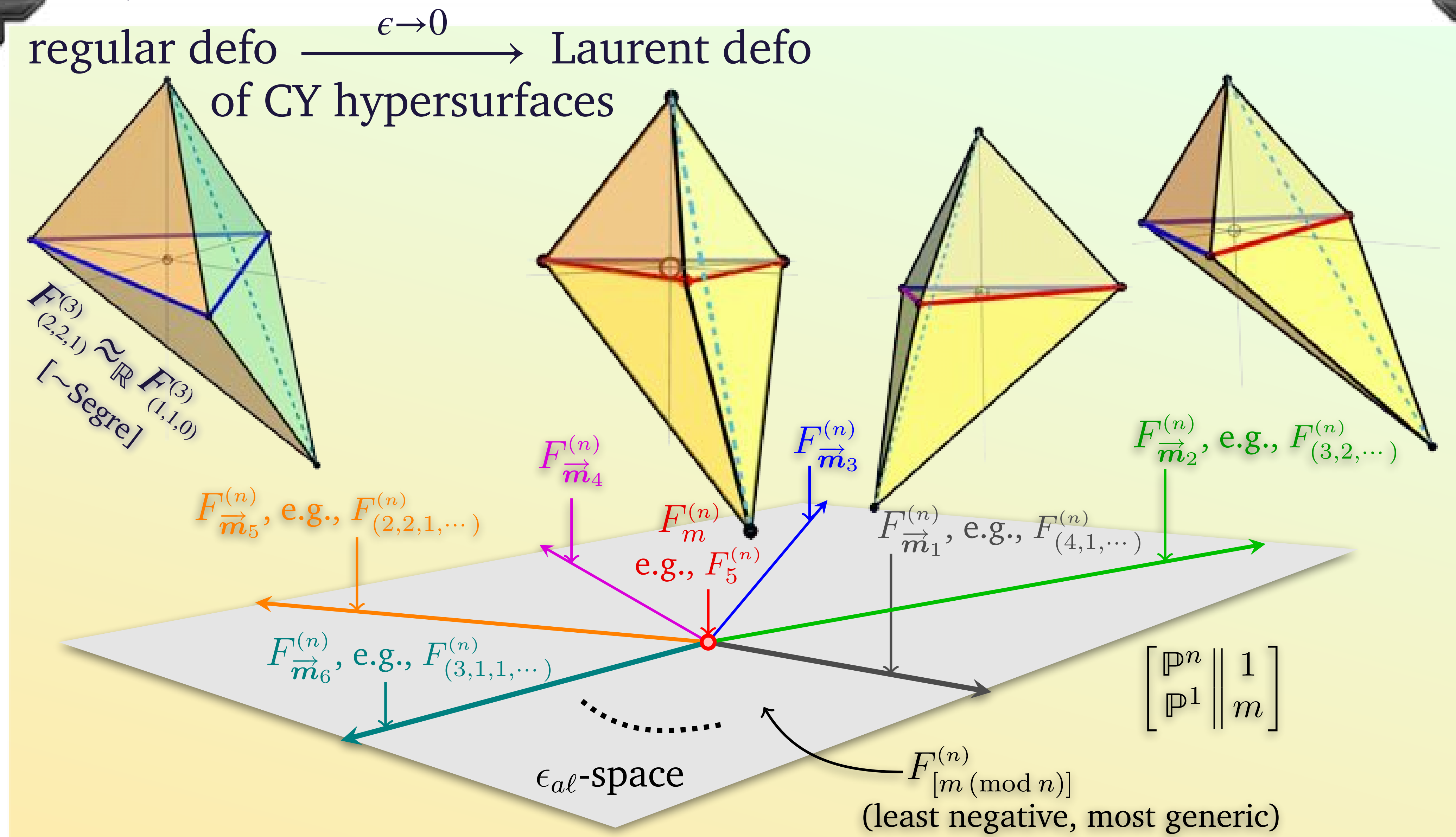
$d(\theta^{(k)}) := k!\,\mathrm{Vol}(\theta^{(k)})$ [BH: *signed* by orientation!]

Laurent GLSM Coda

Summary

—*Proof-of-Concept*—

- CY(n−1)-folds in Hirzebruch n-folds
- Oriented polytopes



A deformation family picture

istr.

$\star_X)^\nabla$

B³H²K mirrors

e

es

on

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ks to be

xtended

Thank You!

https://tristan.nfshost.com/

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