Quantum cohomology of blowups: a conjecture

2020年10月7日 21:30

Online AG seminar at Nottingham

Handout for this talk: https://www.math.kyoto-u.ac.jp/~iritani/talk_Nottingham.pdf

X : smooth projective variety

$QH(X) = (H^*(X), *_\tau)_{\tau \in H^*(X)}$: quantum cohomology

family of comm. rings

$$(\alpha *_\tau \beta, \gamma) = \sum_{n=0}^{\infty} \langle \alpha, \beta, \gamma, \overbrace{\tau, \dots, \tau}^{n} \rangle_{0,n+3,d} \frac{1}{n!}$$

$(\ ,\ )$: Poincaré pairing $\quad d \in H_2(X, \mathbb{Z})$

$*_\tau \longrightarrow \cup$ as $\tau \in H^2(X)$, $\mathrm{Re}\left(\int_d \tau\right) \to -\infty$

(large radius limit) $\quad \forall d$: effective curve $\neq 0$

e.g. … ω ample, …

<u>Crepant transformation conj</u> (Y. Ruan)

$\phi : X_1 \dashrightarrow X_2$ birational

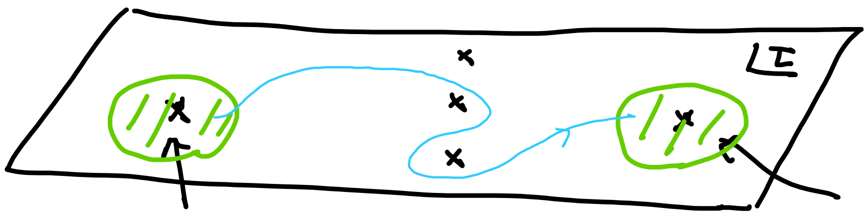
is called <u>crepant</u> if

$$\begin{array}{ccc} & \exists\, Z & \leftarrow \text{sm proj} \\ \exists f \swarrow & & \searrow \exists g \\ X_1 & \overset{\phi}{\dashrightarrow} & X_2 \end{array}$$

f, g : birational morphism

$$f^* K_{X_1} = g^* K_{X_2}$$

$\Rightarrow$ $QH^*(X_1) \cong QH^*(X_2)$ after analytic conti in $\mathbb{C}$



large radius

large radius limit for X_1

large radius limit for X_2

Discrepant transformation Conjecture (?)

$f^* K_{X_1} \leq g^* K_{X_2}$ ($g^* K_{X_2} - f^* K_{X_1}$ is an effective div)

$\Rightarrow$ $QH^*(X_1)$ is a direct summand of $QH^*(X_2)$ (as a ring)

Quantum connection (Dubrovin conn) : conn on the vector bdle

$H \times \mathbb{C}_z \longrightarrow \mathbb{C}_z$

$H := H^*(X)$

$$\nabla^{(\tau)}_{\frac{\partial}{\partial z}} := \frac{\partial}{\partial z} - \frac{1}{z^2}(E * \tau) + \frac{1}{z}\mu$$

- $E = c_1(X) + \sum_i \left(1 - \frac{1}{2}\deg\phi_i\right) \tau^i \frac{\partial}{\partial \tau^i}$ $\qquad \{\phi_i\}$ basis of H

$\tau = \sum_i \tau^i \phi_i$

- $\mu \in \mathrm{End}(H^*(X))$ grading op $\qquad \mu(\phi_i) = \left(\frac{1}{2}\deg\phi_i - \frac{n}{2}\right)\phi_i$

Dubrovin $\qquad \nabla^{(\tau)}_{\frac{\partial}{\partial z}}$: isomonodromic deformation

$$H \times (H_\tau \times \mathbb{C}_z) \to H_\tau \times \mathbb{C}_z$$

- $\nabla^{(\tau)}$: regular sing at $z = \infty$ $\leftarrow$ $\nabla_{\frac{\partial}{\partial z^{-1}}} = \frac{\partial}{\partial z^{-1}} + E *_\tau - \frac{\mu}{z^{-1}}$

irregular sing at $z = 0$

(order 2 pole)

- $\nabla^{(\tau)}$ is self-dual w.r.t Poincaré pairing between fibers at z and $-z$

Conjecture $QC(X)_\tau := \left(H^\times \mathbb{C}_z \to \mathbb{C}_z, \nabla^{(\tau)}_{\partial/\partial z} \right)$

① (formal decomposition) $\overline{QC}(X)_\tau := QC(X)_\tau \otimes_{\mathbb{C}[z]} \mathbb{C}[[z]]$

$$\overline{QC}(X)_\tau \cong \bigoplus_{u \in \mathrm{Spec}(E*_\tau)} \left(e^{u/z} \otimes F_u \right) \otimes_{\mathbb{C}\{z\}} \mathbb{C}[[z]]$$

the set of eigenvalues of $E*$

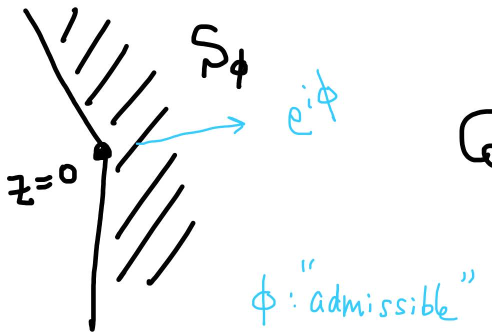
rank 1 conn $(\mathbb{C}\{z\}, d + d(u/z))$

free $\mathbb{C}\{z\}$-module with reg sing conn

[HT thm : in general need ramification $w \longmapsto z = w^r$]

② (analytic lift) **Fact**

the above decomp lifts uniquely to an analytic decomp over a sector of angle $> \pi$ (centered around $e^{i\phi}$)



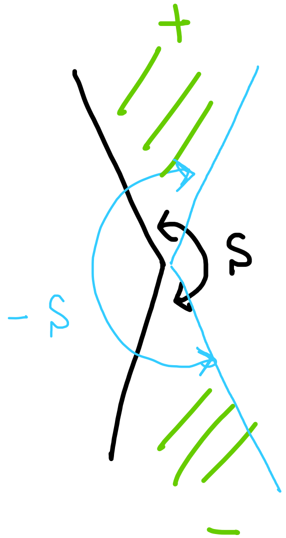
$$QC(X)_z\Big|_S \cong \bigoplus_u e^{u/z} \otimes \mathcal{F}_u\Big|_S$$

③ (SOD and Stokes data)

semi-orthogonal decomposition

V_S : space of flat sections over a sector S

Fact



$$[s_1, s_2) := (s_1(e^{-\pi i} z),\ s_2(z))$$

Poincaré

$$V_S \cong \bigoplus_u V_u \quad \text{s.t.} \quad [V_{u_1}, V_{u_2}) = 0$$

if $\mathrm{Im}(u_1/e^{i\phi}) < \mathrm{Im}(u_2/e^{i\phi})$

$t_+ \downarrow \quad \downarrow t_-$: analytic conti maps

$$V_{-S} \cong \bigoplus V_u'$$

$t_\pm$: Stokes matrix

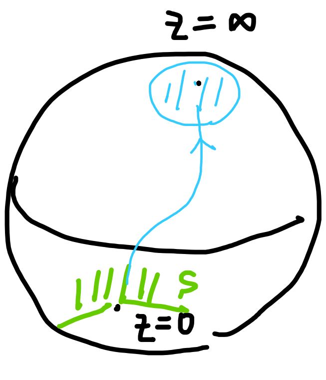
↑ determined by $[\ ,\ )$ and the SOD

$\left\{ \begin{array}{l} \text{formal decomp} \\ \quad + \\ \text{Stokes data} \end{array} \right.$ ⇝ germ at $z=0$ of $\nabla^{(\tau)}$

④ (Dubrovin / Gamma Conjecture)

Galkin - Golyshev - I.

Sanda - Shamoto



H

via analytic continuation to $z=\infty$

the above SOD induces an SOD of top K-grp

$$K_{top}(X) \longrightarrow H(X,\mathbb{C}) \cong \{\text{flat sections near } \infty\}$$

$$\hat{\Gamma}_X \cdot (2\pi i)^{\deg/2}\, ch(\cdot)$$

∃ SOD/$\mathbb{Z}$ ↗

Euler pairing

Blowup $Z \subset X$: codim c $\overbrace{\text{sub variety}}^{\text{smooth}}$

$$\begin{array}{ccc} E & \overset{j}{\hookrightarrow} & \tilde{X} \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ Z & \hookrightarrow & X \end{array}$$

$$\tilde{X} = Bl_Z X$$

Orlov SOD : $D^b(\tilde{X}) = \langle D^b(Z)_{-(c-1)}, \ldots, D^b(Z)_{-1}, D^b(X) \rangle$

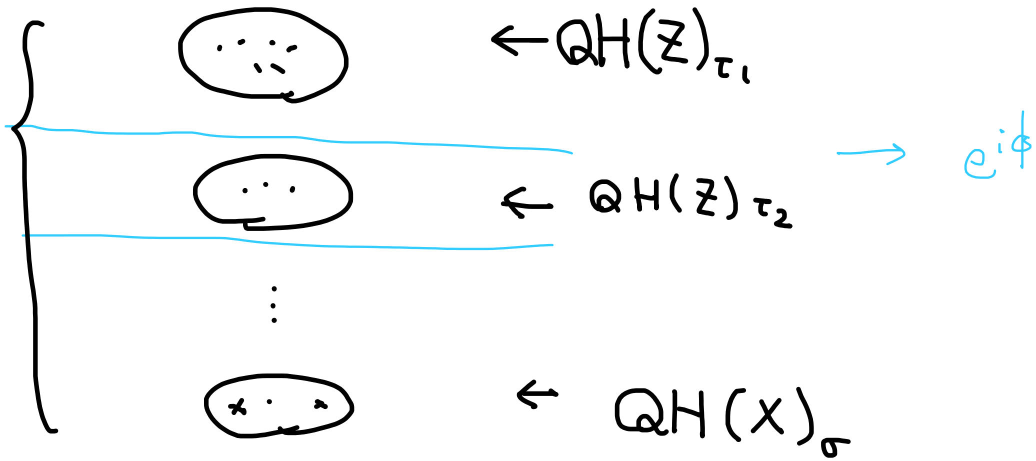
$$D^b(Z)_k = \operatorname{Im}\left(j_* \left(\mathcal{O}(k) \otimes \pi^*(?) \right) \right)$$

- reconstruct $QC(\tilde{X})$ from $QC(X)$ and $QC(Z)$

① $\overline{QC}(\tilde{X}) := \overline{QC}(Z)_{\tau_1} \oplus \cdot \oplus \overline{QC}(Z)_{\tau_{c-1}} \oplus \overline{QC}(X)_\sigma$

$$\tau_i \in H^*(Z) \qquad \sigma \in H^*(X)$$

Suppose $\mathrm{Spec}(E^{Z}*_{\tau_1})$, $\mathrm{Spec}(E^{X}*_{\sigma})$ align as



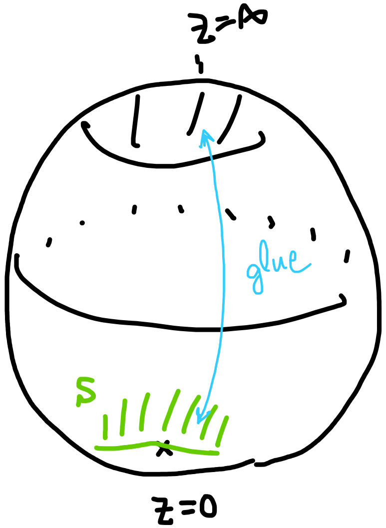
② using Orlov's SOD, we can reconstruct the Stokes str of $QC(\tilde{X})$ from those of $QC(X)$, $QC(Z)$

$\rightsquigarrow$ germ of conn at $z=0$

③ we glue it with the germ of conn near $z=\infty$

$\nabla^{(\tau)}_{z} \sim \boxed{\frac{\partial}{\quad} - c_1(\tilde{X}) + \tilde{\mu}} \leftarrow z=\infty$

$\partial/\partial z$ ... $\frac{\partial}{\partial z}$ ... $\frac{c_1(X)}{z^2}$ ... $\frac{}{z}$



$S(\tau, z)$ → also get a τ for $QC(\tilde{X})$

gauge transf.

↑

calibration

$$\tau = f(\tau_1, \dots, \tau_{c-1}, \sigma)$$

$$H(z)^{c-1} \times H(X)$$
$$\downarrow f$$
$$H(\tilde{X})$$

local isom of F-manifolds

$X = \mathbb{P}^4$ $\quad$ $\tilde{X} = Bl_{\mathbb{P}^1} \mathbb{P}^4$

$\varphi : \tilde{X} \to X$

$\pi : \tilde{X} \to \mathbb{P}^2$, $\mathbb{P}^2$-bdle

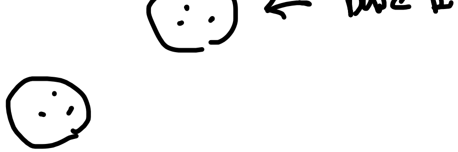
$$\tau = p_1 \log q_1 + p_2 \log q_2$$

$$p_1 = \pi^* H$$
$$p_2 = \varphi^* H$$

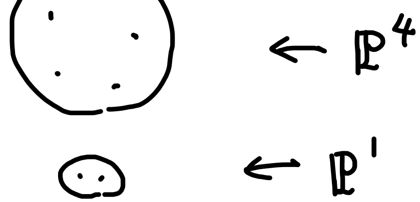
$|q_1| \ll |q_2| \ll 1$

"large base limit"

$|q_2| \ll |q_1| \ll 1$



fibration picture



blow down picture