The moduli continuity method for log Fano pairs

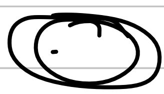
(joint with P. Gallardo and C. Spotti)

1. Intro.

Moduli problem: Describe compactification of some families of log Fano pairs with geometric meaning (compact moduli)

GIT → Good: easy to describe (at least with equations for each variety), construct, classify.

GIT → Bad: hard/never modular.

$\overline{M}^{GIT.}_{C_4 \subseteq \mathbb{P}^2} \supset$  $\times 2$

2 GIT Hypersurfaces

$$D = \{f_d = 0\} \subseteq \mathbb{P}^n \rightsquigarrow [f] \in \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))) = \mathcal{H}$$

Notice $SL_{n+1} \curvearrowright \mathbb{P}^n \rightsquigarrow SL_{n+1} \curvearrowright \mathcal{H}$.

GIT gives a ~~[illegible]~~ compactification of the space of smooth hypersurfaces

$$\overline{M}^{GIT} = \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}(d)))^{ss} /\!/ SL_{n+1} =$$

$$= \operatorname{Proj}\left(\bigoplus_{m \geq 0} H^0(\mathcal{H}, \mathcal{O}(m))^{SL_{n+1}}\right)$$

= closed "semistable" orbits with finite stabilisers

Well known class for (n,d) small

$(n,d) \in \{(2,3), (2,4), (2,5), (2,6),$
$(3,3), (3,4), (3,5)$ partially Gallardo
$(4,3), (5,3)\}$

In GIT semistable orbits are detected by Hilbert-Mumford weight of 1-PS acting on points $p \in \mathcal{H}$.

D (semi)stable $\iff HM(\lambda, D) > 0$ ≥ 0

$\forall \lambda$ 1-PS

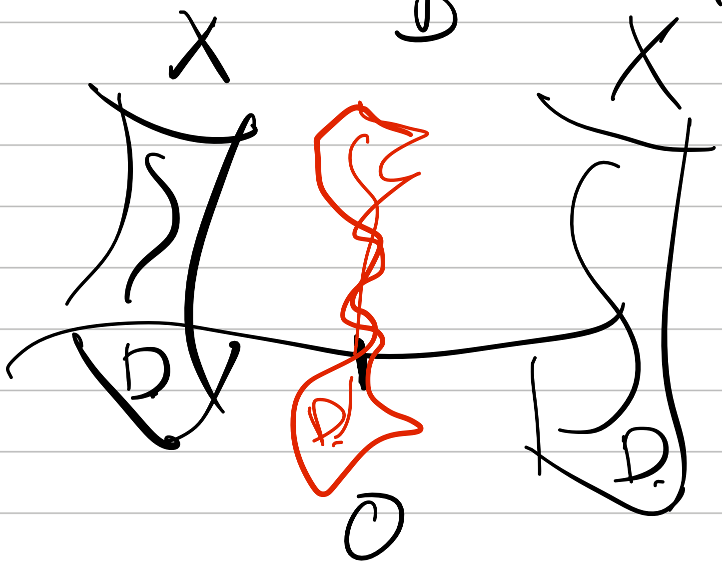
$\lambda : \mathbb{C}^* \longrightarrow SL_{n+1}$

3 K-Stability Let $(X, (1-\beta)D)$ log Fano pair $(-K_X - (1-\beta)D) > 0$.

$\beta \in (0, 1)$.

A test conf. of $(X, (1-\beta)D)$ is a $\mathbb{C}^*$-lcy pair $(\mathcal{X}, \mathcal{D}) \xrightarrow{\pi} \mathbb{C}$ flat proper $\mathbb{C}^*$-equivariant morphism st $(\mathcal{X}|_b, \mathcal{D}_b) \simeq (X, D)$ for $b \neq 0$

and $\mathcal{D}|_b$ is equidimensional.



<u>Note</u> $(X, D) \subseteq \mathbb{P}^N$

$\lambda : \underset{\tau}{\mathbb{C}^*} \longrightarrow SL_{N+1}$

induces a T.C

by taking $\overline{\lambda \cdot (X, D)}\Big|_{\mathbb{C}}$

Not all T.C arise this way.

b/c we cannot have a global $\mathbb{P}^N$

where all T.C live in $(\mathcal{X}, \mathcal{D}) \subseteq \mathbb{P}^N \times \mathbb{P}^1$

<u>Donaldson – Futaki</u>

$DF_\beta(\mathcal{X}, \mathcal{D})$ depending on action on central fibre and Hilbert polynomial

$\underline{\beta}$

$\mathcal{X}$.

DEF $(X,(1-\beta)D)$ is

K-stable $\iff \mathrm{DF}_{\beta}>0$ $\forall$ non-trivial T.C

$\Downarrow$

K-polyst. $\iff \mathrm{DF}_{\beta}\geq 0$ $\forall$ non-equiv. Trivial T.C

$\Downarrow$

K-ss $\iff \mathrm{DF}_{\beta}\geq 0$ $\forall$ T.C.

$\mathrm{Bl}_{\mathcal{J}}\, X\times\mathbb{C}$
$\mathcal{J}\subseteq X_0$.

FACT
If $(\mathcal{X},\mathcal{D})$ is induced from a 1-PS $\lambda:\mathbb{C}^*\longrightarrow SL_{N+1}$.

$$\mathrm{DF}(\mathcal{X},\mathcal{D})=\text{K-HM}(\lambda,D)$$

$D\subseteq X=\mathbb{P}^n$.

CDS + others X is a smooth Fano

X is K-polyst $\iff$ X admits a KE metric

THM (Odaka, Li-Wang-Xu)

$\mathbb{Q}$-Gorenstein smoothable K-polyst. Fano varieties form a compact moduli space.

Ex (Odaka-Spotti-Sun: Description for dP (all).

Spotti-Sun: dP variety of deg = 4.

S-S + ~~L~~X: cubic 3-folds.

→ Moduli continuity method.

Gap Conjecture. (Spotti-Sun)

If p is KLT sing (not smooth) of an n-dim variety W.

$$\widehat{\mathrm{vol}}_{W,p} \leq 2(n-1)^n.$$

THM (Gallardo–MG–Spotti '18)

Let $d \geq n+1$. Assuming G.C. $\exists\, 0 < \beta_0 < 1$

$\forall\, \beta \in (\beta_0, 1)$ the K-polyst compactification $\overline{M}^K$ of pairs $(\mathbb{P}^n, (1-\beta)D)$ is canonically isom. to $\overline{M}_d^{GIT}$.

$D = \{f_d = 0\}$.

CONJ (G–MG–S) X is K-polyst. Fano variety, for sufficiently divisible and large $l \in \mathbb{N}$ $\exists\, \beta_0 = \beta_0(X, l)$ st moduli $\overline{M}^K$ K-polystable pairs $(X, (1-\beta)D)$ is canonically isom to $\mathbb{P}(H^0(X, -lK_X))^{ss} /\!\!/ \mathrm{AuT}(X)$

$D \in |-lK_X|$

Rmks

- Point of Conj: One can always compactify a smoothable family of pairs $(X, (1-\beta)D_i)$ of K-pst pairs but this may be by adding pairs $(X', (1-\beta)D_i')$ with $X' \neq X$.

- Further anecdotical evidence some sing cubic surfaces and $l = 1$

- Thm (w/o ~~CG~~ C assumption) later proven by A-DF-L '19.

Proof Idea Moduli continuity method. (First used by O-S-S '14)

(0) Have a compactification that we "understand" (know everything inside)

For us: $\overline{M}^{GIT}_{X_d \subseteq \mathbb{P}^n} =: \overline{M}^{GIT}$.

(1) $(X, (1-\beta)D) \in \overline{M}^{K} \Longrightarrow [(X, (1-\beta)D)]$
$\cap$
$\overline{M}^{GIT}$.

For us. $X \simeq \mathbb{P}^n$ $D = \{f_d = 0\}$.

(Use: G.C + CDS + Kobayashi-Ochiai)

$\Longrightarrow \exists \phi : \cancel{\overline{M}^{K}}\ \overline{M}^{GH} \longrightarrow \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}(d)))$
$\shortparallel$
$\mathcal{H}$.

(2) Recall $DF(\mathcal{X}, \mathcal{D}) = HM(\lambda, D)$
if $\underset{\smile}{\lambda}$ induces $\underline{\underline{\mathcal{X}}}$.

$\Longrightarrow (\mathbb{P}^n, (1-\beta)D)$, $D \in \mathcal{H}$.
is K-p.s.t. $\Longrightarrow [D]$ GIT
(polystable).

$\Longrightarrow \exists \phi : \cancel{\overline{M}^{K}}\ \overline{M}^{GH} \longrightarrow \mathcal{H}^{ss} /\!\!/ SL_{n+1} = \overline{M}^{GIT}$

(3) $\underbrace{\phi : \overline{M}^{GH}} \longrightarrow \overline{M}^{GIT}$.

Need to show ϕ is homeom.

Properties ϕ:

(a) injective (by uniqueness of KE metrics)

(b) Continuous (Luna slice Thm + CDS).

(c) Im(ϕ) open and close.

$(X, (1-\beta)D)$ log smooth. $\Rightarrow$ KE.

$\Rightarrow$ D smooth. $\overline{M}^{GIT}$.

(d) ~~$\overline{M}^{GIT}$~~ Recall $\overline{M}^{GH}$ compact

$\overline{M}^{GIT}$ Hausdorff.

$\Downarrow$

ϕ is a homeom. ~~$\square$~~

$\Downarrow$

Im(ϕ) compact and close

$\Downarrow$

ϕ is surj.

Bonus Track

~~Fr...~~

$X = \{f_3 = 0\} \subseteq \mathbb{P}^3$.

$D \in |-K_X|$

$D = X \cap H$. if $H \not\subset X$.

Gallardo - MG $\rightsquigarrow$ 11 compactif.

using GIT of pairs (X, D).

In reality GIT $(X, H) \cong (f_3 = 0, l_1 = 0)$

Using same method, if $1 > \beta > \beta_0 = \frac{\sqrt{3}}{2}$
we showed $\overline{M}^K_\beta \cong \overline{M}^{GIT}_1$

(with Spotti).

$(X, D = X \cap H)$ GIT is for pairs (X, H)

$(X, H) \in \overline{M}^{GIT} \Longrightarrow H \not\subset X$.

(X, D)

$\lambda : \mathbb{C}^* \longrightarrow SL_4 \curvearrowright \mathbb{P}^3.$

$$\lambda(t)(X, H) \xrightarrow{t \to 0} (X', H')$$

why should $H' \cap X'$ be a divisor.

in principle it may happen that

$H' \subseteq X'$.

X is a $\mathbb{Q}$-Go. smooth. Fano.

X admits a KE metric $\iff$ X is K-pst.

KE metric with conical sing of angle $2\pi\beta$.

along $D. \overset{loc}{=} \{z_1 = 0\}$.

$$|g_\beta| = |dz_1|^2 |z_1|^{2(\beta-1)} + \sum |dz_i|^2.$$

$$\begin{tikzcd}
\overline{M}_{13}^{GH} \arrow[r, phantom, "\neq"] \arrow[d] & \overline{M}_{13}^{K} \\
\overline{M}^{GIT} &
\end{tikzcd}$$