Combinatorial aspects of graph potencials, and the combinatorial non-abelian Torelli theorem.

(joint with Swarnava M. and Pieter B.)

graph potentials (a little bit more recent than arXiv)

mccme.ru/~galkin/gp.pdf

graph potentials

index.pdf 2 - nc. Torelli

In AG classical (abelian) Torelli theorem.

C = closed Riemann surface genus $g \geq 0$

(smooth proj. alg. curve)

$(J(C), \Theta)$ — Jacobian variety

$\underline{\underline{\mathbb{C}^g/\mathbb{Z}^{2g}}} \leftarrow \dfrac{\text{hol. forms}}{\text{periods}}$

$C \xrightarrow[\sim]{f} C'$ $\quad J(C) \xrightarrow[\sim]{J(f)} J(C')$

Θ-divisor $\rightarrow$ P.P.A.V.

1) Curve C can be algorithmically reconstructed from $(J(C), \Theta)$

2) All isomorphisms defined $(J(C), \Theta) \rightarrow (J(C'), \Theta')$ have geom. origin.

$J \Leftarrow$ Picard variety

Moduli space of rank 1 vector bundles.

(abelian str. group)

Moduli space of rank $r > 1$ vector bundles.

(non-abelian str. group)

(eq. classes of semi-stable)

$$\mathcal{U}_r(C) \xrightarrow{\det} \mathcal{U}_1(C) = \operatorname{Pic} C$$

$$\mathcal{L} \ni \mathcal{L}$$

$$E \longrightarrow \det E$$

fiber $\det^{-1}(\mathcal{L})$

$$\|$$

$$S\mathcal{U}_r(C, \mathcal{L}) \qquad E \to E \otimes \mathcal{M}$$

m.s. of s.s. v.b. with fixed rank r

and $\det E = \mathcal{L}$

$\deg \mathcal{L} \in \mathbb{Z}/r\mathbb{Z}$

what matters.

If $r=2$ $\rightarrow$ even (e.g. $\mathcal{O} = \mathcal{L}$)

$\rightarrow$ odd case.

Var $SU_r(C, \mathcal{L})$ come

with (anti-) canonical polarization

Thm ("non-abelian Torelli" of Kouvidakis-Pantev)

① Curve C + smooth proj

can be uniquely (algorithm.) reconstructed from

$SU_r(C, \mathcal{L})$

② $\mathrm{Aut}\, SU_r(C, \mathcal{L})$

in terms of $\mathrm{Aut}\, C$

and $\mathrm{Pic}(\Sigma_r)$

(1+2) ← Torelli package u.

$\left(\text{Curves}, \; \right) \xrightarrow{\;\mathcal{M}\;}$ Varieties

(C, L)

Curve + bun.

$\boxed{\text{Moduli functor is full.}}$

Combinatorial versions

Curve C ⟶ Graph G

(line bun L) + coloring c.

Moduli Space of Bun. ⟶ Polytope

$U_r(C)$ $\quad P(G)$

$SU_r(C,L)$ $\quad P(G,c)$

Moduli functor

Graphs $\longrightarrow$ Polytopes.

some construction, "natural"

$$G \longrightarrow P(G)$$

① would be an algorithm to go from $P(G)$ to G

② $\mathrm{Isom}(P(G), P(G'))$

Pol. $\wr\wr$

$\mathrm{Isom}_{\text{col. Graphs}}(G, G')$

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abelian comb. torelli

Oda, Artamkin,

Caporaso-Viviani, ...

Graphs G – 3-valent
(possibly with leaves)
abelian: flow polytope.
(dim $= g$)
like J.

Our non-abelian version of Torelli

dim $= 3g-3+\dots$

$\boxed{r=2}$

$$G \longrightarrow P(G)$$

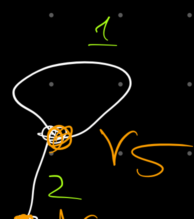
From G to $P(G)$
3-valent

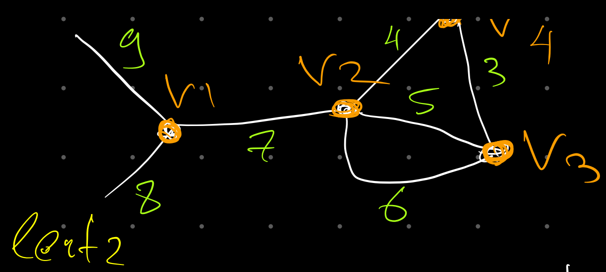
$V(G)$ – vertices

$E(G)$ – (semi-) edges

$\mathbb{R}^E$ affin spac

leaf 2

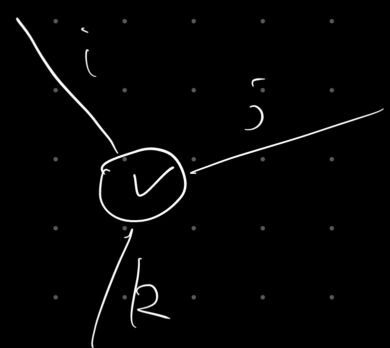




$\langle e(i) \rangle = \mathbb{R}^9$

i-th edge.

every vertex $\to$ 4 vectors



$$\pm e(i) \pm e(j) \pm e(k)$$

$p^{(v,s)}$ 8 Kombinationen.

$\downarrow$ choose 4

$$s(i)e(i) + s(j)e(j) + s(k)e(k)$$

$$s(\cdot) \in \{\pm 1\}$$

$$s(i)\cdot s(j)\cdot s(k) = +1$$

$$s \in \mathbb{F}_2^3$$

$$P(G) := \left\langle \begin{array}{c} p(v,s), \\ v \in V(G) \\ s \text{ - admissible signs} \end{array} \right\rangle$$

$$\overline{\#} = 4 \cdot |V(G)|$$

Def ! of $\boxed{\text{dual quantum Clebsch-Gordan polytope.}}$

Assumptions on G:

1. in every conn. com. ≥ 5 vert.
2. no loops
3. no double edges
4. no leaves.

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under these assumption

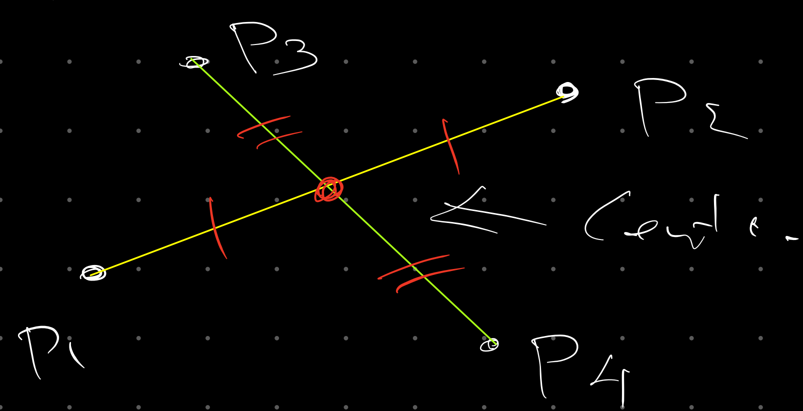
there is a (simple)
algorithm
that reconstructs G
from $P(G)$

Notation $\text{Ray} = \text{Vertex}(P(G))$

using affine rel. between the rays.

0. Center of mass of all the rays equals $0 \in \mathbb{R}^n$

1. How to get the base $\langle \pm e(i) \rangle$



Consider all pairs of diagonals s.t. their convex hull is a parallelogram.

Record their centers.

$\boxed{\text{Claim. The set of these centers equals } \{\pm e(i)\}_{i\text{-edge.}}}$

2. Given a ray P $P = \sum a(i) e(i)$
look for its coordinates
in base $\{\pm e(i)\}$

$\forall P \to \#\{i \text{ s.t. } a(i) \neq 0\} = 3$

use these to define vertices.

Now we have the G

No loops $\Rightarrow$ ans $p(v,s)$
is $(\pm 1, \pm 1, \pm 1, 0, \dots)$
up to perm.

$\sum x_i^2$ ($\pm e(i)$ are o.n.b.)

$\| p(v,s) \|^2 = 3$

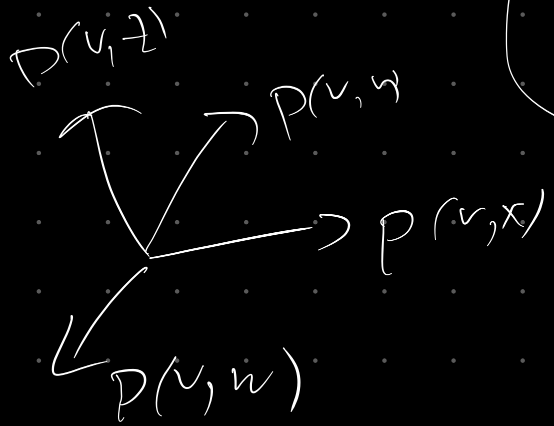
"

every $p(r,s)$ is a ray
(a vertex of $P(G)$)

$v \in V(G)$ $\quad x, y, z, w$

4 signs

+	+	+
+	−	−
−	+	−
−	−	+



$$p(v,x)+p(v,y)+p(v,z)+p(v,w)=0.$$

$$p(v_1,x)+p(v_2,y)=p(v_3,z)+p(v_4,w)$$

T mod 2 is a eu.

$\mathbb{Z}^E = \{\pm e(i)\}$ mod 2.

$f(\ldots)=0$ mi c

$\pm e(i), \pm e(i) \pm e(h)$

$$p(v,x)+p(v,y) = p(v',z)+p(v',w)$$

$\shortparallel$ $\qquad\qquad \Rightarrow \quad v \overset{}{\underset{i}{\text{———}}} v'$ $\qquad \shortparallel$

$\pm e(i)$ $\qquad\qquad\qquad\qquad\qquad\qquad \pm e(i)$

$$p(v,x)-p(v,y) = p(v',z)-p(v',w)$$

$\shortparallel$ $\qquad\qquad\qquad\qquad\qquad \shortparallel$

$\pm e(i)\pm e(j)$ $\qquad\qquad\qquad \pm e(i)\pm e(j)$

$v \underset{j}{\overset{i}{=\!=\!=}} v'$

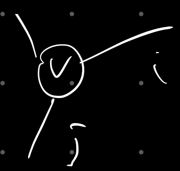
$$G \longleftarrow P(G)$$

Colorings

$$c: V(G) \to \{\pm 1\}$$

$$P(G,c) = \sum_{s(i)s(j)s(k)=c(v)} p(v,s)$$

$\in \mathbb{R}$



Groupoid of colored graphs. $(G,c)\to(G',c')$

① Generators of 1 type

$$f: G \to G'$$

isom of graphs

s.t. $c'\circ f = c$

② if $f = Id_G$,

$$c'/c = \partial b$$

$$b \in C^1(G, \partial G, \dots)$$

$$c' = c + \delta(i)$$

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f - isom of graphs

+ 1-cochain b s.t.

$$\partial b = c' - f^{*} c$$

<u>Theorem</u> (Colored combinatorial non-abelian $(r=2)$ Torelli theorem)

$$P : (G, c) \to P(G, c)$$

is a <u>**full**</u> functor

from the groupoid of colored graphs
to the groupoid of affine real (convex) polytopes

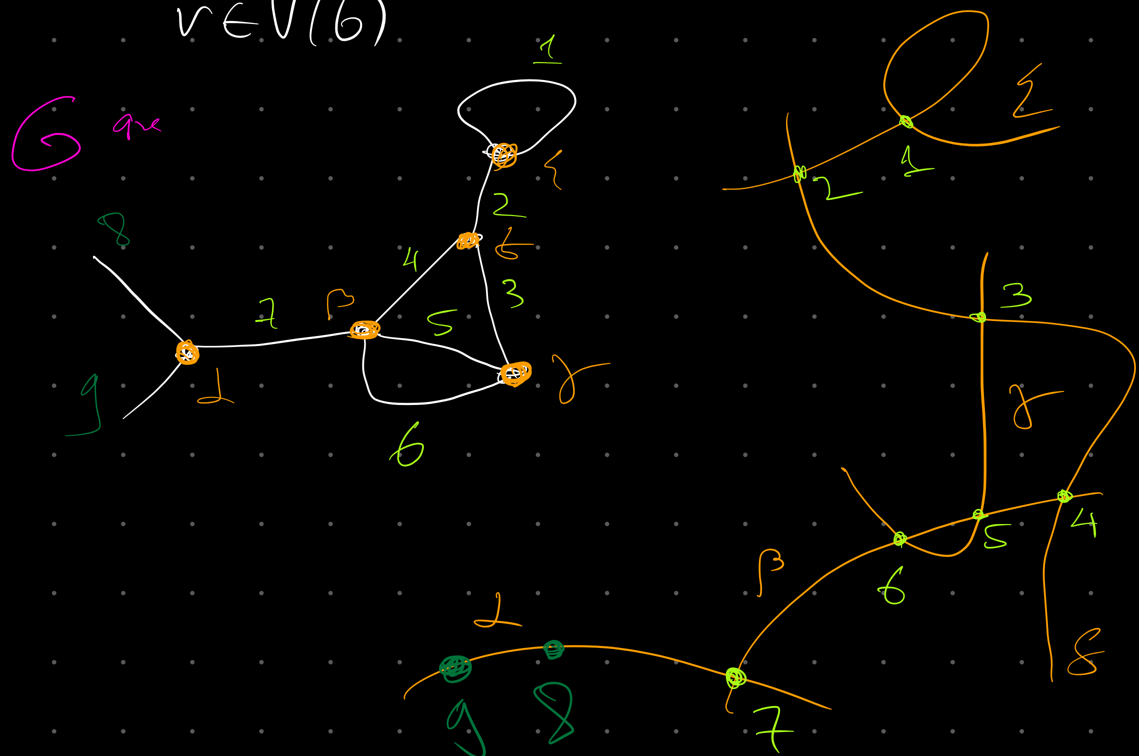
Curves Graphs.

"Graph Curves"

$$\bigcup \mathbb{P}^1_r$$

$v \in V(G)$

G graph



$C(G)$ given curve.

marked stable curve at the deepest corner of Deligne-Mumford moduli space.

Degeneration

$$C_{smooth} \longrightarrow C_G$$

Vanishing Cycles = Thursto

Cecil system

Moduli Spaces

$\mathrm{Pic}\, C$

and $SU_r(C, L)$

also vary with singular curve C.

$r=1$ $\longrightarrow$ Normalization of compactified Jacobian of stable curves are toric varieties

flow polytopes.

$r=2$ case.

$\mathrm{Pic}\ SU_r(C,L) = \mathbb{Z}\,\Theta$

(4) $\Gamma\big(SU_2(C,L),\ \mathcal{O}(n\cdot\Theta)\big)$

n

vector bundle

over $\mathcal{M}_{g,n}$

< Faltings ...

(4) Conformal blocks

in SU(2) WZW model

CFT

constructed by TUY (1990)

Tsuchiya-Ueno-Yamada

They constructed sheaves

over DM comp.

$\overline{\mathcal{M}_{g,n}}$

~2009 C. Manon.

Naturally extends to a sheaf of graded algebras.

Proj (. . .)

$\top$ Flat Proj. Family of varieties over

$M_{g,n} \subset \overline{M}_{g,n}$

$\cup\!\!\!\!\!\!\!\!\quad$ (C)

$\uparrow$

$SU_2(C,L)$

It turns out, that

for $\underline{\underline{r=2}}$ over the

generic curves, Mukai varieties

are toric^{Fan} varieti

$\nearrow\!\!\swarrow$

Polytope $\underline{P(6,6)}$

quantum Clebsch-Gordan

θ's

(Same as in the
work of Jeffrey-Weitsman)

Our $^{\text{combi.}}$ Non-abelian Torelli
theorem has
2 interesting
applications.

① Symplectic Geometry

② Theory of Random Walks.

① $\mathcal{N}_g$ — moduli space of (twisted) flat $SU(2)$ conn. on Σ_g.

$$SU_2''(C, L)$$

$\deg L$ - odd

C - smooth genus g.

$\forall$

$$G \longmapsto L(G) \subset \mathcal{N}_g$$

Say ~~e~~ → Monotone Lagrangian torus

~~Thm~~ ⇐ Corollary

~~Conj~~ $L(G)$ is Hamiltonian-isotopic to $L(G') \Leftrightarrow G = G'$

Thm Conj is true for G and G' that $X_G, X_{G'}$ have small resolution of sing.

Small Resolution Conjecture

G - connected graph $\quad g \geq 2$

no leaves

Toric Variety $\quad d$ - odd.

$X_{P(G,C)}$ has a small resolution of singularities

$\iff$ graph G is 3-connected

i.e. G is still connected if one removes any 2 edge

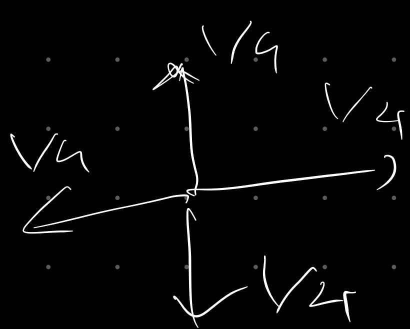
Q: Can you hear the shape of a random walk?

$v_1, \dots, v_N \in \mathbb{Z}^n \subset \mathbb{R}^n$

$p_1, \dots, p_N \in \mathbb{R}_{\geq 0}$

$$\boxed{p_1 + \dots + p_N = 1}$$

$$p_1 v_1 + \dots + p_N v_N = 0 \in \mathbb{R}^n$$



$(1,0)$	$1/4$
$(0,1)$	$1/4$
$(-1,0)$	$1/4$
$(0,-1)$	$1/4$

Shape $= \langle v_i \rangle$

Convex hull
of V_i

Hearing.

$P(n)$ = probability to return to 0 (or x)
in n steps.

Seq. of numbers.

"Period Sequence".

Q: Given $P(n)$
can you recover
the shape?

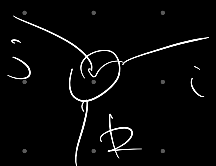
Answer: NO

$(G, c) \to$ Random Walk

$V_i \to P(v,s)^{4k}$

Random walk $\Longleftrightarrow$ Laurent Pols.

$$W_{G,c}(z) := \sum_{v,\, s \in AC(v,i,j,k)} \frac{1}{4\#V(G)}\, z_i^{s(i)}\, z_j^{s(j)}\, z_k^{s(k)}$$



$$\mathbb{R}[\mathbb{Z}^E] = \mathbb{R}[z_1^{\pm 1}, \dots, z_{|E|}^{\pm 1}]$$

$$P(n) = \frac{1}{(2\pi i)^{\#E(G)}} \int_{|z_i| = R_i} W^n \, d\log z_1 \wedge \dots \wedge d\log z_n$$

$$= \text{c.t.}(W^n)$$

<u>Claim</u> $P_{G,c}(n)$

depends only on homotopy type of (G, c)

(g_i, n_i) $\underline{\underline{\sum c_i \bmod 2}}$

<u>Proof</u> Formula for change of coord. in the integral

<u>Reason</u> $W_\gamma = f(x, y, z)$

$$f(x,y,z) = \frac{xy}{z} + \frac{xz}{y} + \frac{yz}{x} + \frac{1}{xyz}$$

$$f_f(x,y,z) = \hat{p}\, xyz + \frac{x}{yz} + \frac{y}{xz} + \frac{z}{yx}$$

$p=1 \to$ quantity

$p=0 \to$ classically

This function

satisfies is $\mathbb{Z}_3$-sym and

functional equation.

$$f(x,y,s)+f(z,w,s) = f(x,z,t)+f(y,w,t)$$

Rir WDVV ЭНХ.

$$t = f(x,y,z,w;s)$$

s.t. it is volume-pres

$$\frac{dx}{x}\frac{dy}{y}\frac{dz}{z}\frac{dw}{w}\left(\frac{ds}{s}-\frac{dt}{t}\right)=0$$

$$W_{G,c} = \sum_{v} f_{c(v)}(z_i,z_j,z_k)$$

T

Def. of Graph Potential.

$$F_{G,c}(z_1, \dots, z_n; t) \quad \langle \mathrm{coeff}_{t^n}(F) = P(n)$$

leaf vo

$$= \int \exp(t \cdot W_{G,c}(z))$$

<u>Thm</u> $F_{G,c}(z_1 \dots t)$ depends only on hom. type of (G, c)

$$F_g(z_1, \dots, z_n)$$

Collection of $F_{g,n}$ $g \geq 0$, $n \dots$

$$2g - 2 + n > 0$$

is 2d. TQFT

on Hilbert Space $L^2(S^1)$

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the change of coord. $s \to t$ $g=0$, $n > \ldots$

is some "quantum"

deformation of

Plücker formular

for $Gr(4,2)$ GL model

Grass Part $\longrightarrow$ Laurent poly (R,W)

Graphs $\xleftarrow{\text{Torelli}}$ $\downarrow$ Newton.

(quantum) $\longrightarrow$ Polytopes

C Grad...

Graph Curves $\xrightarrow{\text{Using Abou. ... ... }}$ Toric Varieties

Mirror Symmetry

de

Smo Curve $\xleftarrow{\text{n.a. Torelli}}$ Moduli Space

$SU_2(C, L) = N_g$

M

Fam info

$\overline{\overline{SG \text{ of } T}}$

„n.a.-Torelli conjecture"
(interpolation between our comb. thm and e.g. KP)

$\forall$ stable curve $C \in \overline{M}_{g,n}$

Manon/Tuy variety associated to C recovers C.