MATRIX FACTORIZATIONS FOR DISCRIMINANTS OF PSEUDO-REFLECTION GROUPS

This a talk about a long running project about a McKay correspondence for reflection groups that was started with Ragnar BUCHWEITZ and Colin INGALLS in 2014.

⤳ see arXiv: 1709.04218

Focus of this talk : pseudo-reflection groups and recent work of my student Simon MAY about the family of complex reflection groups $G(r,p,n)$, and work in progress with Ingalls, May, Talarico about explicit MF's for S_n.

⤳ see arXiv 2107.12196

Plan of talk:
- I McKay correspondence
- II Matrix factorizations + MCM modules
- III Pseudo-reflection groups
- IV Isotypical components of S/z
- V Dimension 2, the groups $G(r,p,n)$

Overall goal of our work: Find a McKay correspondence for finite reflection groups $G \subset GL(n,\mathbb{C})$: $G \circlearrowright \mathbb{C}^n$

$\{\text{irreps of } G\} \xleftrightarrow{1-1}$ Geometry on $\mathbb{C}^n/G$

$\longleftrightarrow$ Certain modules on invariant ring

I McKay correspondence

for talk

Classical setup: Let $G \subset SL(2,\mathbb{C})$ finite subgroup

G acts on $\mathbb{C}^2$ and on $S := \mathbb{C}[x,y]$ by: $g \in G$: $g \cdot f = f(g(x,y))$, $f \in S$

with invariant ring $R := S^G = \{ f \in S : g \cdot f = f \}$.

Quotient singularities $\mathbb{C}^2/G := \mathrm{Spec}(R)$: these are surface singularities (embedded in $\mathbb{C}^3$) with isolated sing at. 0, classified by ADE diagrams. [Klein 1884 Du Val 1934])

dual resolution graphs

J. McKay observerved 1979 a (surprising) direct relation between geometry of $\mathbb{C}^2/G$ and rep. theory of G:

$\{\text{exc. curves on } \widetilde{\mathbb{C}^2/G}\} \xleftrightarrow{1-1} \{\text{irreps of } G\}$

$\widetilde{\mathbb{C}^2/G}$ = min. res. of sing

via $\boxed{\text{dual res graph}} \longleftrightarrow \boxed{\text{McKay quiver of } G \setminus \text{triv.}}$ MK(G)

③

rep

Dynkin diag, type ADE

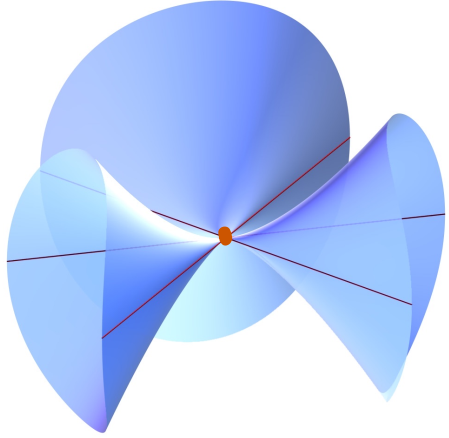
constructed from irreps of G

$MK(G)$ = ext. Dynkin quiver of type ADE

Example $G = D_4 = \langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \rangle$

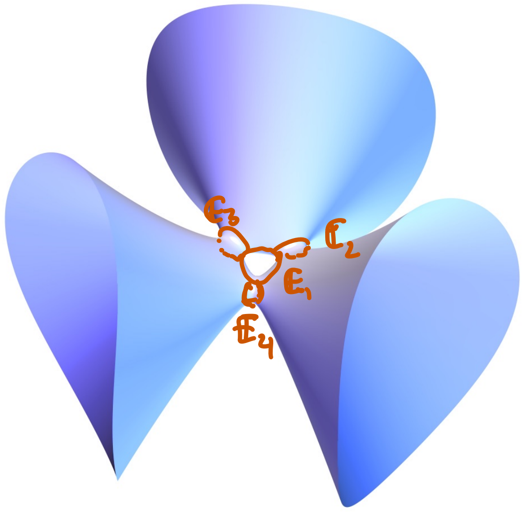
$\mathbb{C}^2/G = \operatorname{Spec}(\mathbb{C}[X,Y,Z]/(Z^2 + YX^2 + 4Y^3))$

Real pic of $\mathbb{C}^2/G$

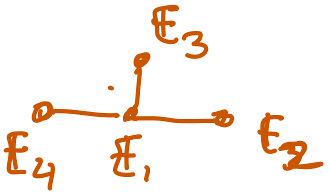


$\xleftarrow{\pi}$

Pic of min. res. $\widetilde{\mathbb{C}^2/G}$

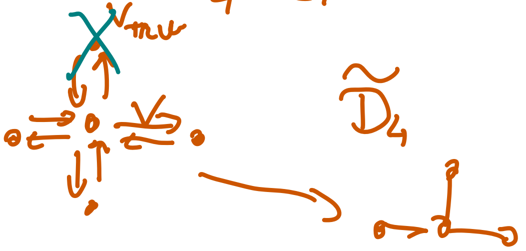


Dual resolution graph:



D_4

$MK(G) =$



$\widetilde{D}_4$

Correspondence more conceptually explained by [Gonzales-Sprinberg – Verdier 1983] and many others [Esnault 1985], [Knörrer 1985], [Herzog 1978] [Artin-Verdier 1985]

In the 2000s renewed interest:

[Kapranov-Vasserot 2000]: Derived equivalence

$$D^b(\mathrm{mod}_G\, \mathbb{C}[x,y]) \simeq D^b(\mathrm{Coh}(\mathbb{C}^2/G))$$

Generalization to higher dim ($G \subseteq SL_3(\mathbb{C})$):

[Bridgeland-King-Reid 2001]: Equiv. of crepant res.

There is in particular Auslander's algebraic version of the correspondence:

Thm [Auslander 1986] $G \subseteq SL(2,\mathbb{C})$

$$\begin{tikzcd} \{\text{irreps of } G\} \arrow[r, leftrightarrow, "1-1"] \arrow[d, leftrightarrow, "1-1"'] & \{\text{isom. classes of MCM-modules over } R\} \arrow[d] \\ \{\text{isom. classes of direct summands of } S \text{ as } R\text{-module}\} & \text{want to understand } MCM(R)! \end{tikzcd}$$

via $S*G \cong \mathrm{End}_R S$

And $MK(G) =$ AR-quiver of $MCM(R)$

Rmk $S*G \cong \mathrm{End}_R(S)$ is a NCCR of $R = S^G$.

Rmk Auslander's correspondence holds more generally for $G \subset GL(n,\mathbb{C})$ small (i.e. G does not contain any pseudo-reflections).

Many generalizations of McKay corr: [Buchweitz 2012] overview

↳ all for small subgrps

<u>Question</u>: What about reflection groups G?

<u>II Matrix factorizations</u> (very short intro!)

A = polyn. ring (graded) / reg. local complete, $f \neq 0 \in A$

$A/(f)$ hypersurface ring

<u>Def</u> A MF of f is a pair of $n \times n$ matrices (M, N) over A, s.t. $M \cdot N = N \cdot M = \mathbb{1}_n \cdot f$.

Every MF defines a MCM-module on $A/(f)$:

$$0 \to A^n \xrightarrow{M} A^n \to \underset{\in MCM(A/(f)).}{\boxed{\operatorname{coker}(M)}} \to 0$$

<u>Thm</u> [Eisenbud 1980]: If (A, m) is reg. local / A graded poly., then have equiv. of categories:

$$\underline{MCM}(A/(f)) \simeq \underline{RMF}(f) \quad \text{``}MF(f)/\{(1,f);(f,1)\}\text{''}$$

$$\wr / [\text{Buchweitz}]$$

$$D^b_{sing}(A/(f)) = D^b(\text{mod } A/(f)) / \text{Perf}(A/(f))$$

<u>So</u>: Understand MCM's over hypersurface ring $\longleftrightarrow$ understand MFs!

<u>III Pseudo-reflection groups</u> (=c. refl. groups)

$G \underset{\text{finite}}{\leq} GL(V)$ is <u>true reflection group</u> if G is generated by reflections of order 2.

$V \cong \mathbb{C}^n$

$G \curvearrowright \mathbb{C}^n$

<u>pseudo-refl. grp</u> if generated

Classification: $G(r,p,n)$ + 34 exceptional cases. by cx. reflections

- $S = \mathrm{Sym}_{\mathbb{C}} V \cong \mathbb{C}[x_1, \dots, x_n]$, assume G acts linearly on S.

- $R = S^G$ = invariant ring

By [Chevalley-Shephard-Todd] $R \cong \underset{\nearrow \text{ basic invariants}}{\mathbb{C}[f_1, \dots, f_n]}$ $\swarrow$ polyn. ring

$\Rightarrow \mathbb{C}^n/G = \mathrm{Spec}(R)$ is smooth for G p.-r. group.

But: have to look at codim 1:

- $\mathcal{A}(G) \subseteq V$ reflection arrangement = set of mirrors def. by poly $\underset{S}{\overset{}{J}} \in S$, $J = \prod_{s \in \mathrm{Refl}(G)} \ell_s^{m_s - 1}$ and $z = \prod_{s \in \mathrm{Ref}(G)} \ell_s$ $\swarrow$ reduced

If G is true refl. group: $m_s = 2$, and $J = z \in S$.

- $V(\Delta) \subseteq V/G$ discriminant of G = image of $\mathcal{A}(G)$ under $\pi : V \to V/G$

def by $\Delta = z \cdot J \in R$.

Ex: $G = S_3 \curvearrowright \mathbb{C}^2$

$S = \mathbb{C}[x,y]$

$R = \mathbb{C}[\underbrace{x^2y + xy^2}_{X}, \underbrace{x^2 + yx + y^2}_{Y}]$

$S/(z) = S/((x+2y)(y+2x)(x-y))$

$R/\Delta = R/(4Y^3 - 27X^2)$

irreps of S_3

⑦

$\mathrm{End}_{R/\Delta}(\boxed{S/z}) \underset{R/\Delta}{\cong} R/\Delta \oplus M \oplus$ $\quad M \cong \mathfrak{m}^2$

Both $V(\Delta)$ and $\mathcal{A}(G)$ hypersurfaces, non-normal, free divisors.

AR-quiver : $$\begin{tikzcd} \underset{R/\Delta}{\bullet} \arrow[r, shift left] & \underset{\mathfrak{m}}{\bullet} \arrow[l, shift left] \arrow[loop right] \end{tikzcd}$$

McKay correspondence for $^{\text{true}}$reflection groups:

[BFI '20] : For true ref. groups get an analogue of Auslander's correspondence:

{irreps of G} \ triv $\xleftrightarrow{1-1}$ {ind. direct summands of S/z as an R/Δ-module}

$S*G =: A$

$$\underset{\substack{\downarrow \\ \text{triv. idempotent}}}{A/AeA} \cong \underbrace{\mathrm{End}_{R/\Delta}(S/z)}_{\text{NCR of } R/\Delta}$$

In part: direct summands of $S/z \longleftrightarrow$ MF's of Δ

$\updownarrow$ 1-1

irreps G

in dim 2: $V(\Delta)$ = ADE-curve sing.
S/z is rep. gen. of $\mathrm{MCM}(R/\Delta)$

Q: (1) What happens if G is gen. by refl. of order ≥ 3?
($\rightsquigarrow A/AeA \not\cong \mathrm{End}_{R/\Delta}(S/z)$)

(2) What is $\mathrm{gldim}_{R/\Delta}(S/z)$ for $n \geq 3$?

IV Isotypical components of S/z (assume G p. refl. group)

S/z → S/J

$$S/z \underset{R/\Delta\text{-mod}}{\cong} \bigoplus_{i=1}^{r} M_i \otimes_{\mathbb{C}} V_i$$

$V_1, \ldots, V_r$ = irreps of G, $M_i \cong \mathrm{Hom}_{\mathbb{C}G}(V_i, S/z)$

Thm [BFI]: Can identify some of the M_i

- V_i = triv $\Rightarrow M_{\text{triv}} = 0$

- $V_i = \det^{-1} \Rightarrow M_{\det^{-1}} = R/\Delta$.
- $V_i = V \Rightarrow M_V \cong \mathrm{Syz}(\mathrm{jac}(\Delta)) \cong \mathrm{Der}(-\log \Delta)$
- $V_i = \Lambda^m V \quad M_{\Lambda^m V} \cong \Lambda^m \mathrm{Der}(-\log \Delta)$

$\mathrm{Syz}(M_{\Lambda^m V}) \cong R^{m-1}$ ← log. residues of Δ.

$G(r,p,n) \rightsquigarrow$ $G(1,1,n) \cong S_n$
$S(2,1,n) \cong (\mathbb{Z}/2) \wr S_n$

Thm [May]: $G = G(r,p,2)$, then can describe all isotypical comp. of $S/z \rightsquigarrow S/z$ is still a rep. generator for $R/\Delta \Rightarrow \mathrm{End}_{R/\Delta}(S/z)$ is a NCR of R/Δ.

But: $\mathrm{End}_{R/\Delta}(S/z) \underset{\text{Morita}}{\not\simeq} A/AeA$

Proof: Use higher Specht polynomials + compute MF's of R/Δ.